Spin glass dynamics below T_g

Chapter 2.3 in "Spin Glasses and related problems",
Vik.Dotsenko, M. Feigel'man and L.Ioffe (1990)

1. Dynamic approach to SK spin glass - reminder

2. General equation for Slow Cooling (L.Ioffe, Phys.Rev. B**38**, 5181, 1988, also ZhETF **93**, 343, 1987)

3. Solution of the SC eqs near T_c, comparison with the RSB results

4. Simplest non-trivial example of history-dependent non-ergodic response

Relaxational dynamics of the Edwards-Anderson model and the mean-field theory of spin-glasses

H. Sompolinsky & Annette Zippelius   Phys. Rev. B 25, 6860 (1982)

Order parameter: $$q_{\mathrm{EA}} = \lim_{t\to\infty} [\langle S_i(0) S_i(t) \rangle]_J \ .$$

The EA Hamiltonian is $$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \ ,$$ spin variables S_i take the values ± 1

Distribution of random J_{ij}

$$P(J_{ij}) = (2\pi z/\tilde{J}^2)^{-1/2} \times \exp[-z(J_{ij} - J_0/z)^2 / 2\tilde{J}^2]$$

Z is the number of nearest neighbours

We consider here a soft-spin version of the EA model defined by

$$\beta H = \frac{1}{2} \sum_{\langle ij \rangle} (r_0 \delta_{ij} - 2\beta J_{ij}) \sigma_i \sigma_j + u \sum_i \sigma_i^4 + \sum_i h_i \sigma_i, \quad \beta = 1/T \ .$$

To study the relaxational dynamics of spin glasses, we propose a simple phenomenological Langevin equation,

$$\Gamma_0^{-1}\partial_t\sigma_i(t) = -\frac{\delta(\beta H)}{\delta\sigma_i(t)} + \xi_i(t)$$
$$= \sum_j (r_0\delta_{ij} - \beta J_{ij})\sigma_j(t)$$
$$+ 4u\sigma_i^3(t) + h_i(t) + \xi_i(t) .$$

$$\langle \xi_i(t)\xi_j(t')\rangle = \frac{2}{\Gamma_0}\delta_{ij}\delta(t-t') ,$$

Objects of interest: pair spin correlation function

$$C_{ij}(t-t') = \langle \sigma_i(t)\sigma_j(t')\rangle$$

and the linear-response function

$$G_{ij}(t-t') = \frac{\partial\langle \sigma_i(t)\rangle}{\partial h_j(t')}, \quad t > t'$$

The FDT reads in the present context

$$C_{ij}(\omega) = \frac{2}{\omega}\mathrm{Im}G_{ij}(\omega)$$

and

$$C_{ij}(t=0) = G_{ij}(\omega=0)$$

$$\mathrm{Re}G_{ij}(\omega) = -\int \frac{d\omega'}{\pi}\frac{\mathrm{Im}G_{ij}(\omega')}{\omega-\omega'}$$

Dynamic generating functional:

P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1978).
C. De Dominicis, J. Phys. (Paris) C 1, 247 (1976); C. De Dominicis and L. Peliti, Phys. Rev. B 18, 353 (1978).

$$Z\{J_{ij},l_i,\hat{l}_i\}=\int D\sigma\, D\hat{\sigma}\exp\left[\int dt\, l_i(t)\sigma_i(t)+i\hat{l}_i(t)\hat{\sigma}_i(t)+L\{\sigma,\hat{\sigma}\}\right]$$

$$L\{\sigma,\hat{\sigma}\}=\int dt\sum_i i\hat{\sigma}_i(t)\left[-\Gamma_0^{-1}\partial_t\sigma_i(t)-r_0\sigma_i(t)+\beta\sum_j J_{ij}\sigma_j(t)-4u\sigma_i^3(t)-h_i(t)+\Gamma_0^{-1}i\hat{\sigma}_i(t)\right]+V\{\sigma\}$$

The term V, which arises from the functional bian, is given by[32,33]

and ensures the proper normalization of Z,

$$Z\{J_{ij},l_i=\hat{l}_i=0\}=1$$

$$V=-\frac{1}{2}\int dt\sum_i\frac{\delta^2(\beta H)}{\delta\sigma_i^2}\quad=-\int dt\sum_i[\tfrac{1}{2}r_0+6u\sigma_i^2(t)]$$

$$\left.\frac{\delta^n\delta^m\ln Z}{\delta\hat{l}_1(\hat{t}_1)\cdots\delta l_m(t_m)}\right|_{l_i=\hat{l}_i=0}\quad=\langle i\hat{\sigma}_1(\hat{t}_1)\cdots\sigma_m(t_m)\rangle_c$$

Response function: $\langle i\hat{\sigma}_j(t')\sigma_i(t)\rangle=G_{ij}(t-t')$ (t > t')

Averaging over J_{ij} is possible since Z = 1

$$[Z]_J \equiv \int \prod dJ_{ij} P(J_{ij}) Z\{J_{ij}\} = \int D\sigma D\hat{\sigma} \exp \Bigg| L_0\{\sigma,\hat{\sigma}\} + \frac{\beta J_0}{z} \sum_{\langle ij \rangle} \int dt\, i\hat{\sigma}_i(t)\sigma_j(t)$$

$$+2\frac{\beta^2 \tilde{J}^2}{z} \sum_{\langle ij \rangle} \int dt\, dt' [i\hat{\sigma}_i(t)\sigma_j(t') i\hat{\sigma}_i(t')\sigma_j(t) + i\hat{\sigma}_i(t)\sigma_j(t) i\hat{\sigma}_j(t')\sigma_i(t')] \Bigg|$$

here

we use the property $J_{ij} = J_{ji}$.

$$L_0\{\sigma,\hat{\sigma}\} = \int dt \sum_i [i\hat{\sigma}_i(-\Gamma_0^{-1}\partial_t\sigma_i - r_0\sigma_i - 4u\sigma_i^3 - h_i + i\Gamma_0^{-1}\hat{\sigma}_i) + V\{\sigma\} + i\hat{l}_i\hat{\sigma}_i + l_i\sigma_i]$$

Decoupling of the 4-th order terms:

$$[Z]_J = \int \prod_{\alpha}^{4} DQ_\alpha^i(t,t') \exp\Bigg(-\frac{z}{\beta^2\tilde{J}^2} \int dt\, dt' \sum_{i,j} (K^{-1})_{ij} [Q_1^i(t,t')Q_2^j(t,t') + Q_3^i(t,t')Q_4^j(t,t')]$$

$$+\ln \int D\sigma D\hat{\sigma} \exp L\{\sigma,\hat{\sigma},Q_\alpha\} \Bigg|,$$

where K is the short-range matrix ($K_{ij}=1$ if i,j are nearest neighbors and zero otherwise), and

$$L\{\sigma,\hat{\sigma},Q_\alpha\} = L_0\{\sigma,\hat{\sigma}\} + \frac{1}{2} \int dt\, dt' \sum_i [Q_1^i(t,t') i\hat{\sigma}_i(t) i\hat{\sigma}_i(t') + Q_2^i(t,t')\sigma_i(t)\sigma_i(t')$$

$$+ Q_3^i(t,t') i\hat{\sigma}_i(t)\sigma_i(t') + Q_4^i(t,t') i\hat{\sigma}_i(t')\sigma_i(t)].$$

(We have assumed $J_0 = 0$.)

$$L\{\sigma_i\hat{\sigma}_i\}=L_0\{\sigma_i,\hat{\sigma}_i\}+\frac{\beta^2\tilde{J}^2}{2}\int dt\,dt'[\,C(t-t')i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')+2G(t-t')i\hat{\sigma}_i(t)\sigma_i(t')]\;,$$

$$C(t-t')\equiv[\langle\sigma_i(t)\sigma_i(t')\rangle]_J\;,$$

$$G(t-t')\equiv[\langle i\hat{\sigma}_i(t')\sigma_i(t)\rangle]_J$$

$$L_0\{\sigma,\hat{\sigma}\}=\int dt\sum_i[i\hat{\sigma}_i(-\Gamma_0^{-1}\partial_t\sigma_i-r_0\sigma_i-4u\sigma_i^3-h_i+i\Gamma_0^{-1}\hat{\sigma}_i)+V\{\sigma\}+i\hat{l}_i\hat{\sigma}_i+l_i\sigma_i]$$

The new effective bare propagator is

$$G_0^{-1}(\omega)=r_0-i\omega\Gamma_0^{-1}-\beta^2\tilde{J}^2G(\omega)$$

and the effective noise ϕ is a Gaussian random variable with width

$$\langle\phi_i(\omega)\phi_i(\omega')\rangle=[2\Gamma_0^{-1}+\beta^2\tilde{J}^2C(\omega)]\,\delta(\omega+\omega')$$

$$\sigma_i(\omega)=G_0(\omega)[\phi_i(\omega)+h_i(\omega)]$$
$$-4uG_0(\omega)\int d\omega_1d\omega_2\sigma_i(\omega_1)\sigma_i(\omega_2)$$
$$\times\sigma_i(\omega-\omega_1-\omega_2)\;.$$

DYNAMICS FOR $T \geq T_c$

Key quantity: $$\Gamma^{-1}(\omega) = i\frac{\partial G^{-1}(\omega)}{\partial \omega}$$ Effective kinetic coefficient

The dynamic-response function obeys a Dyson equation

$$G^{-1}(\omega) = G_0^{-1}(\omega) + \Sigma(\omega) \ ,$$

$$G_0^{-1}(\omega) = r_0 - i\omega\Gamma_0^{-1} - \beta^2 \tilde{J}^2 G(\omega)$$

the nonlinear coupling u.

$$\Gamma^{-1}(\omega) = \frac{\Gamma_0^{-1} + i\frac{\partial \Sigma}{\partial \omega}}{1 - \beta^2 \tilde{J}^2 G^2(\omega)}$$

random exchange

assume that $$\frac{\partial}{\partial \omega}\operatorname{Im}\Sigma(0) \equiv \frac{\partial}{\partial \omega}\operatorname{Im}\Sigma\bigg|_{\omega=0}$$ is finite (it will be proven later)

Then $$\Gamma^{-1}(\omega=0) = \frac{\Gamma_0^{-1} + \operatorname{Im}\frac{\partial \Sigma}{\partial \omega}(0)}{1 - \beta^2 \tilde{J}^2 G^2(0)} \ .$$ diverges at $T_c = \tilde{J} G(0)$

Dynamics below T_c

$$\langle x \rangle_A = \int \mathscr{D}\sigma \, \mathscr{D}\hat{\sigma} \, x(\sigma, \hat{\sigma}) \exp [A(\sigma, \hat{\sigma})], \quad A = A_0 + A_1 + W.$$

$$A_0 = i \int \mathrm{d}t \sum_i \hat{\sigma}_i(t) \Bigg[- \Gamma_0^{-1} \frac{\partial \sigma_i}{\partial t} - r\sigma_i(t) - u\sigma_i^3(t) + \beta h_i(t) + \Gamma_0^{-1} \mathrm{i}\hat{\sigma}(t) \Bigg],$$

$$A_1 = i \int \mathrm{d}t \sum_{i,j} \hat{\sigma}_i(t) \beta J_{ij} \sigma_j(t),$$

$$W = -\frac{1}{2} \int \mathrm{d}t \sum_i \frac{\delta^2 \beta H}{\delta \sigma_i^2} = -\frac{1}{2} \int \mathrm{d}t \sum_i [r_0 + 3u\sigma_i^2(t)].$$

Averaging over random J + time-dependence of T

$$A = A_0 + A_{\text{eff}} + W,$$

$$A_{\text{eff}} = -\frac{1}{2N}\int \mathrm{d}t\,\mathrm{d}t' \sum_{i,j} [\hat{\sigma}_i(t)\hat{\sigma}_i(t')\sigma_j(t)\sigma_j(t')$$

$$+ \hat{\sigma}_i(t)\sigma_i(t')\hat{\sigma}_j(t')\sigma_j(t)](\beta J)_t(\beta J)_{t'},$$

Mean-field approximation:

$$A_{\text{eff}} = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}t' \sum_i [C(t, t')\mathrm{i}\hat{\sigma}_i\mathrm{i}\hat{\sigma}_i\mathrm{i}\hat{\sigma}_i(t') + 2G(t, t')\mathrm{i}\hat{\sigma}_i(t)\sigma_i(t)] \times (\beta J)_t(\beta J)_{t'}.$$

The equivalent problem: Langevin equation with noise

$$\Gamma_0^{-1}\sigma_i^0 = -\frac{\delta H_0}{\partial \sigma_i} + (\beta J)_t \int \mathrm{d}t'\; G(t, t')\sigma(t')(\beta J)_{t'} + \zeta_i(t),$$

$$\langle \zeta_i(t)\zeta_i(t')\rangle = [2\Gamma_0^{-1}\delta(t-t') + (\beta J)_t(\beta J)_{t'}C(t, t')].$$

Fast and slow response and correlations

$$G(t,t') = \Delta(t,t') + \tilde{G}_t(t-t'),$$
$$C(t,t') = q(t,t') + \tilde{C}_t(t-t'),$$

FDT:

$$\tilde{G}_t(t-t') = \tilde{C}_t(t-t').$$

Slow parts   Fast parts

$$\Gamma^{-1}(\omega) = \mathrm{i}\,\frac{\partial \tilde{G}^{-1}(\omega)}{\partial\omega} \qquad \tilde{G}^{-1}(\omega) = \tilde{G}_0^{-1}(\omega) + \Sigma(\omega) \qquad \Gamma^{-1}(\omega) = \frac{\Gamma_0^{-1} + \mathrm{i}\partial\Sigma/\partial\omega}{1-(\beta J)^2\tilde{G}^2(\omega)}$$

Slow part: introduce an auxiliary slowly varying Gaussian variable $Z(t)$

Variance: $\langle Z(t)Z(t')\rangle = (\beta J)_t q(t,t')(\beta J)_{t'}$

$$\Gamma_0^{-1}\dot{\sigma}_i = \left[-\frac{\delta H_0}{\delta\sigma_i} + \beta J\int \mathrm{d}t'\,\tilde{G}_t(t-t')\sigma(t') + \tilde{\xi}(t)\right] + H(t)$$

$$\langle\xi(t)\xi(t')\rangle = 2\Gamma_0^{-1}\delta(t-t') + (\beta J)^2\tilde{D}_t(t-t'),$$

$$H(t) = (\beta J)_t\int \mathrm{d}t'\,\Delta(t,t')\sigma(t')(\beta J)_{t'} + Z(t).$$

From now on, we suppose that the variation of the external parameters is very slow, so that local equilibrium is achieved long before the external parameters vary sufficiently, and the equation (2.3.13) for the fast relaxation can be solved ignoring the variation of the effective field $H(t)$. The presence of a slow part $\Delta(t, t')$ of the response function means that the field produces nonzero magnetization long after it has been switched off ($t-t' \gg \Gamma_0^{-1}$), so we can say that the magnetic field can be frozen into the system.

The perturbation theory in u shows that the "fast" parts of correlators obey the FDT to all orders in u, so we suppose that even below T_c the FDT (2.3.10) holds for the "fast" parts of the correlators. The FDT (2.3.10) implies that at a given value of the effective field $H(t)$

$$\tilde{G}(\omega=0) = \langle\sigma^2\rangle - \langle\sigma(t)\sigma(t')\rangle_{|t-t'|\gg\Gamma_0^{-1}} = 1 - m_t^2, \quad m_t = \langle\sigma_t\rangle .$$

Here and below in this section we denote the average over the fast noise $\tilde{\xi}(t)$ by $\langle . . .\rangle$ and that over the slow noise $Z(t)$ by $[. . .]$, and we consider the Ising normalization of spin lengths: $\langle\sigma^2\rangle = 1$. From (2.3.15), we conclude that the reaction to the slow effective field $H(t)$

$$m_{t_1} = \tanh\{H(t_1) + h(t_1)\}.$$

Fast dynamics on top of frozen magnetization

$$\Gamma^{-1}(\omega) = \frac{\Gamma_0^{-1} + i\,\tilde{G}^{-2}(\omega)\,\partial[\Sigma_H(\omega)\,\tilde{G}_H^2(\omega)]_Z/\partial\omega}{1-(\beta J)^2[\tilde{G}_H^2(\omega)]_Z}$$

$$\Gamma^{-1}(\omega) = \frac{\Gamma_0^{-1} + i\frac{\partial\Sigma}{\partial\omega}}{1-\beta^2\tilde{J}^2G^2(\omega)}$$

Stability condition: $1 - (\beta J)_i^2[\tilde{G}_H^2(0)]_Z \geqslant 0$

via FDT $1 - 2[m_i^2]_Z + [m_i^4]_Z \leqslant \frac{T^2}{J^2}$

In fact it is equality !

Fast response:

$G(\omega) \sim \omega^{\nu}$

Exponent is determined from:

$$\frac{4\pi\coth\pi\nu}{B(\nu,\nu)} = \frac{[2m(1-m)^2]_Z}{[(1-m^2)^3]_Z}$$

(Sompolinsky-Zippelius 1982)

Equations for slow response

$$H(t) = (\beta J)_t \int \mathrm{d}t' \, \Delta(t, t')(\beta J)_t m(t') + Z(t).$$

At

self-consistency equations $\Delta(t_1, t_2) =$ dm(t_1)/dh(t_2) $t_1 - t_2 \gg \Gamma_0^{-1}$

$\partial m_t / \partial(\beta h_t) = 1 - m^2$: $m_{t_1} = \tanh\{H(t_1) + h(t_1)\}.$

$$\Delta(t_1, t_2) = \Big[(1-m^2)_{t_1}(\beta J)_{t_1}\Delta(t_1, t_2)(\beta J)_{t_2}(1-m^2)_{t_2} + (\beta J)_{t_1}(1-m^2)_{t_1}\int \mathrm{d}t \Delta(t_1, t)(\beta J)_t \Delta(t, t_2)\Big]_Z.$$

$$q(t, t') = [m(t)m(t')]_Z.$$

Basic equations for slow cooling description of the SK spin glass

for any nonzero solution $\Delta(t_1, t_2)$, the condition $[(1-m^2)^2]_Z = T^2/J^2$ must be fulfilled, which is exactly the condition for marginal stability

Explicit form of SC Eqs near T_c

In the vicinity of T_c the equations of state are substantially simplified. First, we note that the exact condition for marginal stability determines $q_r = [m_r^2]$ to second order in τ (we use $[m^4] \approx 3q^2 + O(\tau^3)$): $q = \tau + \tau^2$

In the leading order over τ we find equation for anomalous part of response:

$$\Delta(t_1, t_2)\{2q^2(t_1, t_2) - \tau^2(t_1) - \tau^2(t_2)\} + \int \mathrm{d}t\, \Delta(t_1, t)\Delta(t, t_2). = 0$$

The equation for $q(t_1, t_2)$ follows from its definition, i.e. $q(t_1, t_2) = [m(t_1)m(t_2)]$ and (2.3.16). Keeping only the leading terms, we get

$$q(t_1, t_2)\left\{\frac{2}{3}\, q^2(t_1, t_2) - \tau^2(t_1) - \tau^2(t_2)\right\} + \int \mathrm{d}t\, [\Delta(t_1, t)q(t_2, t) + \Delta(t_2, t)q(t_1, t)] + h(t_1)h(t_2) = 0$$

Reparametrization invariance: $\tilde{t} = f(t)$

$$\Delta(t_1, t_2) \rightarrow \tilde{\Delta}(\tilde{t}_1, \tilde{t}_2) = \Delta(t_1, t_2)\, \mathrm{d}t_2/\mathrm{d}\tilde{t}_2$$

Solution for monotonic cooling

In the simplest case of a monotonic decrease of temperature,

$\tau(t) = t$ and h = 0

The solution is:

$$\Delta(t, t') = 2\theta(t-t')t', \quad q(t, t') = \theta(t-t')t' + \theta(t'-t)t$$

It leads to the result for field-cooled susceptibitiy:

$$\chi_{FC} = \int dt' \; T_{t'}^{-1} G(t, t') = T^{-1}(1-q(t, t)) + \int dt' \; \beta_{t'} \Delta(t, t')$$

$$\chi_{ZFC} = T^{-1} \int dt' \; \tilde{G}(t, t') = T^{-1}\{1-q(t, t)\}.$$

$$\chi_{FC} = \chi_{ZFC} + T_c^{-1}\tau^2$$

$$= \{1 + O(\tau^3)\} T_c^{-1}$$

Comparison between SC and RSB approaches

In the leading order over $\tau \ll 1$ the results for physical quantities coincide

Higher-order terms can also be computed in this approach:

$$\left.\begin{aligned} \Delta(t,t') &= 2t' + t'(t-t'), \\ q(t,t') &= t' + \tfrac{1}{2}t'(t+t'), \end{aligned}\right\} \quad t > t'$$

Using the functions $\Delta(t,t')$ and $q(t,t')$ from (2.3.30), and the marginal stability condition (2.3.18), we get the Edwards–Anderson order parameter $q_{\mathrm{EA}} = q(t,t)$ for a state resulting from a slow cooling process to high accuracy in τ:

$$q_{\mathrm{EA}} = \tau + \tau^2 - \tau^3 + \frac{5}{2}\tau^4 - 17.2\tau^5 + O(\tau^6)$$

$$q_{\mathrm{EA}} - q_{\mathrm{eq}} = 2.3\tau^5$$

discrepancy

Internal energy in SC and RSB:

$$U = \frac{1}{2}\sum_{i,j}\langle J_{ij} S_i S_j\rangle \quad = -\frac{1}{2T}\left\{1 - q^2(t,t) + 2T\int \mathrm{d}t'\ \Delta(t,t')q(t,t')\beta(t')\right\}.$$

Marginal stability condition helps to calculate the above integral to high order in τ

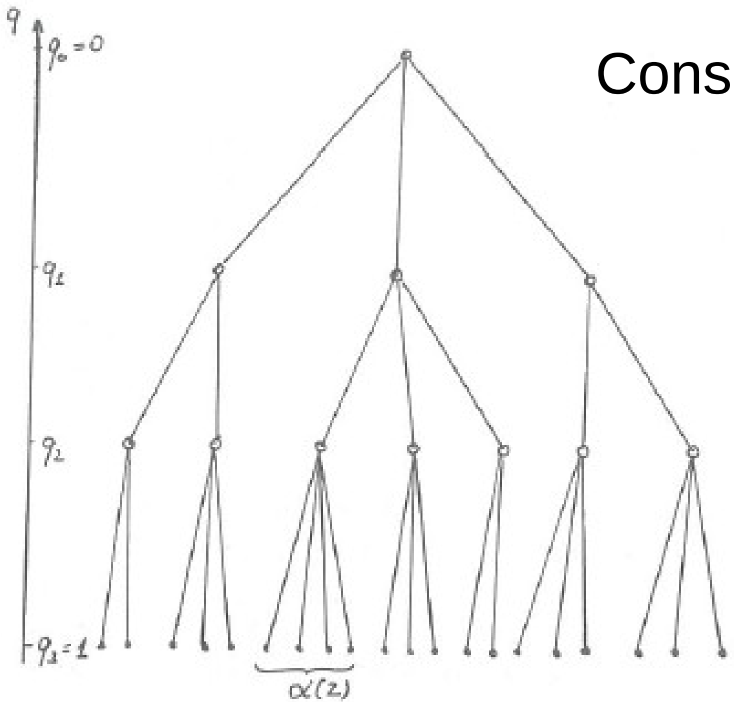
$$U = -\frac{1}{2T}\left(1 - \tau^2 + \frac{2}{3}\tau^2 + \tau^4 - \frac{13}{5}\tau^5\right)$$

This is slightly above the internal energy of the equilibrium state:

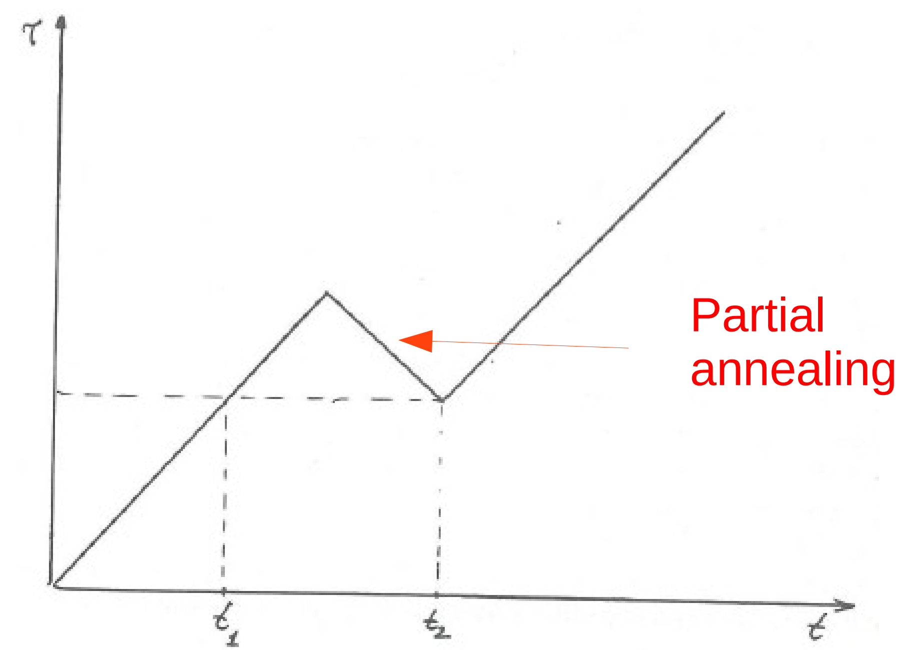
$$U - U_{\mathrm{eq}} = \tfrac{1}{5}\tau^5 + O(\tau^6)$$

SQ process results in a non-equilibrium state. However, probably this state is most relevant in terms of physical observables

How can we see hierarchical nature of SG states ?



Consider non-monotonic variation of temperature:



Thermal history of the SK model considered in the text ($\epsilon = (T_c - T)/T_c$).

How will it affect anomalous response ?

Solution of the SC equations

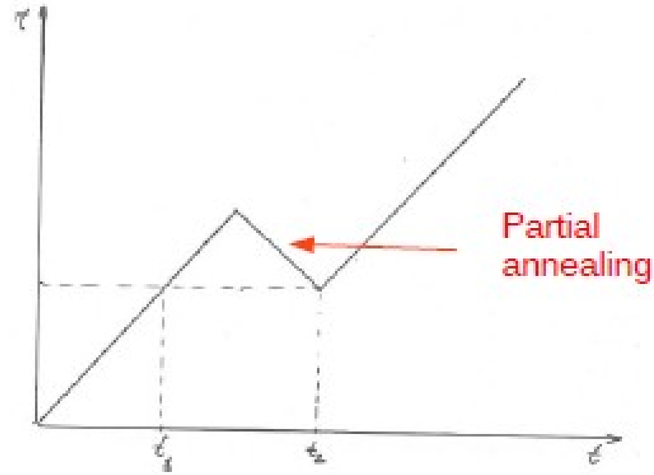
$$\Delta(t_1, t_2)\{2q^2(t_1, t_2) - \tau^2(t_1) - \tau^2(t_2)\} + \int dt\, \Delta(t_1, t)\Delta(t, t_2). = 0$$

$$q(t_1, t_2)\left\{\frac{2}{3} q^2(t_1, t_2) - \tau^2(t_1) - \tau^2(t_2)\right\} + \int dt\, [\Delta(t_1, t)q(t_2, t) + \Delta(t_2, t)q(t_1, t)] + h(t_1)h(t_2) = 0$$

Solution:

$$\Delta(t, t') = \begin{cases} 2t' & (t' \leqslant t_1,\ t' \geqslant t_2), \\ 0 & (t_1 \leqslant t' \leqslant t_2), \end{cases}$$

$$q(t, t') = t',$$



“Memory” was partially erased during annealing

Few problems to solve

- Derive and solve SC equations near T_c for XY spin glass and Heisenberg spin glass (the most interesting quantity to look for is "transverse stiffness").
- Consider slow cooling trajectory on the (T,h) plane of Ising spin glass, of circular form with center at the point $T=T_c(1-\tau)$ where $\tau \ll 1$, and h=0. Assuming that maximal field is << than Almeida-Thouless field $h_{AT} \sim \tau^{3/2}$, try to describe the effect of such rotation in the parameter space.
- Consider "hysteresis loop" due to variation of magnetic field from 0 to $+h_0$ than to $-h_0$. Calculate resulting magnetization.