# Spin glass models with stable 1-step RSB

**THE SIMPLEST SPIN GLASS**

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**SPIN GLASSES WITH *p*-SPIN INTERACTIONS**

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1. Random-energy model (Derrida) as p $\rightarrow\infty$ limit of "p-spin glass": elementary analysis

2. Replica solution in the p $\rightarrow\infty$ limit (Gross & Mezard)

3. Replica solution for general p > 2 (Elizabeth Gardner)

4. Random directed polymer problem (Feigel'man, Ioffe, Mezard 2010)

# The generalized $p$-spin SK model

$$\mathcal{H} = - \sum_{1 \le i_1 < i_2 \ldots < i_p \le N} J_{i_1 i_2 \ldots i_p} \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_p}$$

The interaction strengths are independent random variables which can be taken, for simplicity, to be gaussian. In order for the free energy to be extensive (i.e. proportional to $N$) the probability distribution of the $J$'s must be scaled as follows.

$$P(J_{i_1 \ldots i_p}) = \sqrt{\frac{N^{p-1}}{\pi p!}} \exp\left[ - \frac{(J_{i_1 \ldots i_p})^2 N^{p-1}}{J^2 p!} \right]$$

For $p = 2$ this reduces to the standard SK model. We shall be interested in particular in the $p \to \infty$ limit of these models, where much simplification occurs Note that one must be careful to take the $p \to \infty$ limit *after* taking the thermodynamic limit, $N \to \infty$

B Derrida, Phys Rev Lett 45 (1980) 79, Phys Rev B24 (1981) 2613

Let $\{\sigma_i^{(1)}\}$ denote a given configuration of the spins with energy $\mathscr{H}(\sigma^{(1)})$ This energy depends, of course, on the particular choices of the couplings $J$ The probability, $P(E)$, that it equals $E$ is given by $P(E) = \overline{\delta(E - \mathscr{H}(\sigma^{(1)}))}$, where $\bar{O}(\langle O \rangle)$ stands for the average over the couplings (the thermodynamic average)

$$\overline{O(J, \sigma)} = \int \prod \mathrm{d}J\, P(J) O(J, \sigma) \qquad \langle O(J, \sigma) \rangle = \frac{1}{Z} \sum_{\sigma_i = \pm 1} \mathrm{e}^{-\beta \mathscr{H}(J, \sigma)} O(J, \sigma)$$

Since the $J$ have gaussian distribution, $P(E)$ is easily evaluated in the $N \to \infty$ limit

$$P(E) = \frac{1}{\sqrt{N\pi J^2}} \exp\left[ -\frac{E^2}{J^2 N} \right] \qquad (4)$$

Note that $P(E)$ is independent of $p$ (which justifies the scaling of eq (2)) and of the spin configuration. This is a consequence of "gauge invariance", namely the fact that $\mathscr{H}(\sigma, J) = \mathscr{H}(\sigma', J')$ and $P(J) = P(J')$, where $J'_{i_1 \ldots i_p} = J_{i_1 \ldots i_p} (\sigma_{i_1} \sigma'_{i_1}) \ldots (\sigma_{i_p} \sigma'_{i_p})$

Now consider two different spin configurations, $\{\sigma_i^{(1)}\}$ and $\{\sigma_i^{(2)}\}$ and calculate the probability, $P(E_1, E_2)$, that they have energies $E_1$ and $E_2$ respectively. Due to the gauge invariance this can only depend on the *overlap*, $q$, between the two configurations

$$q^{(1,2)} \equiv \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(1)} \sigma_i^{(2)}$$

One finds (as $N \to \infty$) $$P(E_1, E_2, q) = \overline{\delta(E_1 - \mathcal{H}(\sigma^1))\delta(E_2 - \mathcal{H}(\sigma^2))}$$

$$= [N\pi J^2(1+q^p)N\pi J^2(1-q^p)]^{-1/2} \exp\left[-\frac{(E_1+E_2)^2}{2N(1+q^p)J^2} - \frac{(E_1-E_2)^2}{2N(1-q^p)J^2}\right]$$

*if* $\sigma^{(1)}$ *and* $\sigma^{(2)}$ *are*

*macroscopically distinguishable* $(|q^{(1,2)}| < 1)$ *the energies are uncorrelated*, namely

$$P(E_1, E_2, q) \xrightarrow{p \to \infty} P(E_1)P(E_2) \quad (|q| < 1)$$

when $q = 1$, $P(E_1, E_2, q) = P(E_1)\ \delta(E_1 - E_2)$

Therefore in the large $-p$ limit the energy levels become independent random variables The physics is identical to that of Derrida's random energy model, defined as a system of $2^N$ independent random energy levels distributed according to eq. (4)

Since the energy levels are independent random variables the average number of levels, $\langle n(E)\rangle$, of energy $E$ is simply the total number of levels, $2^N$, times the probability of finding $E$

$$\langle n(E)\rangle = \frac{1}{\sqrt{\pi N J^2}}\, e^{N[\ln 2-(E/NJ)^2]}$$

If $|E| < E_0 = N\sqrt{\ln 2}$ the average number of levels is very large

the fluctuations are of order $1/\sqrt{\langle n(E)\rangle}$

On the other hand if $|E| > E_0$ there are

simply no levels (with probability one) Therefore the entropy is

$$S(E) = N\left[\ln 2 - \left(\frac{E}{NJ}\right)^2\right], \quad |E| < E_0$$

Using $dS/dE = 1/T$ one finds that the free energy is

$$\frac{F}{N} = \begin{cases} -T\ln 2 - J^2/4T, & T > T_c \\ -\sqrt{\ln 2}, & T < T_c \end{cases}$$

The critical temperature, $T_c$, is

$$T_c = 1/(2\sqrt{\ln 2})$$

Below $T_c$ the system gets stuck in the lowest available energy level, $E = -E_0$ and the entropy vanishes. Having completely disposed with the spin configurations, it is not easily seen that this model describes a spin glass Some evidence is provided by the behaviour of the magnetic susceptibility below $T_c$, which can be derived by similar arguments [8]. In the following we shall solve the $p \to \infty$ SK model directly and the spin glass nature of the low-temperature phase will be more apparent

# Replica solution for infinite-p limit

Overlap: $$q^{\alpha\beta}=\frac{1}{N}\sum_{i=1}^{N} m_i^{\alpha} m_i^{\beta}$$

Overlap distribution function $$P(q)\equiv\sum_{\alpha,\beta} P_\alpha P_\beta \delta(q-q^{\alpha\beta})$$

$$Q_{ab}(\sigma)\equiv\frac{1}{N}\sum_{i=1}^{N}\sigma_i^a\sigma_i^b$$

$$\int \overline{P(q)}\,\mathrm{e}^{uq}\,\mathrm{d}q=\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{a\neq b}\mathrm{e}^{u\langle Q_{ab}\rangle},$$

In the $p\to\infty$ model we shall be able to calculate explicitly the function $P(q)$ (this cannot be done in the finite-$p$ case) using the replica method to calculate $\overline{Z^n}$ (we hereafter set $J=1$)

$$\overline{Z^n} = \int \prod \mathrm{d}J_{i_1 \ldots i_p} P(J_{i_1 \ldots i_p}) \times$$

$$\times \ \mathrm{Tr}_{(\sigma_i^a)} \left[ \exp \beta \sum_{a=1}^{n} \left[ \sum_{i_1 < \ldots < i_p} J_{i_1 - i_p} \sigma_{i_1}^a - \sigma_{i_p}^a + h \sum_i \sigma_i^a \right]\right]$$

$$= \mathrm{Tr}_{(\sigma_i^a)} \exp \left[ \tfrac{1}{4}\beta^2 N \left( n + \sum_{a \neq b} Q_{ab}^p(\sigma) \right) + \beta h \sum_{i,a} \sigma_i^a \right]$$

The spin trace can be performed by constraining $Q_{ab}(\sigma)$ to equal $Q_{ab}$, with the aid of a Lagrange multiplier matrix $\lambda_{ab}$. One then gets

$$\overline{Z^n} = \mathrm{e}^{nN\beta^2/4} \int_{-\infty}^{+\infty} \prod_{a<b} \mathrm{d}\, Q_{ab} \int_{-i\infty}^{+i\infty} \prod_{a<b} \frac{\mathrm{d}\lambda_{ab}}{2\pi} \mathrm{e}^{-NG(Q_{ab}, \lambda_{ab})} \qquad (22)$$

$$G(Q_{ab}, \lambda_{ab}) = -\frac{1}{4}\beta^2 \sum_{a \neq b} Q_{ab}^p + \frac{1}{2} \sum_{a \neq b} \lambda_{ab} Q_{ab}$$

$$- \ln \mathrm{Tr}_{(\sigma_a)} \exp \left[ \frac{1}{2} \sum_{a \neq b} \lambda_{ab} \sigma_a \sigma_b + \beta h \sum_a \sigma_a \right]$$

Unlike the case $p=2$, the effective hamiltonian is <u>not</u> quadratic in $Q_{ab}$, which therefore cannot be eliminated. In the limit $N \to \infty$, $\overline{Z^n}$ is given by the dominant saddle-point of $G$, namely mean field theory is exact, and the average free energy is $+\beta \bar{F}/N = \lim_{n\to 0}[G/n - \frac{1}{4}\beta^2]$ Actually one must find the absolute *maximum* of $G$, not the minimum. This reversal is one of the strange features of the $n \to 0$ limit

Since the matrix of fluctuations (of $Q_{ab}$ or $\lambda_{ab}$) has $\frac{1}{2}n(n-1)$ parameters, it acts, for $n<1$, on a space of negative dimensions. In this situation the role of negative and positive eigenvalues is switched [6] and stability requires that $G$ be maximized!

In order to evaluate $G$ explicitly one must impose some ansatz on the structure of $Q_{ab}$, and a corresponding structure on $\lambda_{ab}$. For example in the high-temperature phase, the replica-symmetric ansatz is reasonable since we expect only one pure state

$$Q_{ab} = Q\,,$$

$$\lambda_{ab} = \lambda, \quad a \neq b.$$

$$\mathrm{D}z \equiv \frac{\mathrm{d}z}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2}$$

$$\frac{1}{n}\,G(Q,\lambda) \overset{n\to 0}{=} \tfrac{1}{4}\beta^2 Q^p - \tfrac{1}{2}\lambda Q - \int_{-\infty}^{+\infty} \mathrm{D}z \ln\left[2\,\mathrm{ch}\,(z\sqrt{\lambda} + \beta h)\right] \qquad (25)$$

The saddle-point equations are

$$\tfrac{1}{2}\beta^2 p Q^{p-1} = \lambda\,, \qquad Q = \int \mathrm{D}z\, \mathrm{th}^2\,(z\sqrt{\lambda} + \beta h)$$

When $p = \infty$ there exists a unique saddle-point for *all* $\beta$, $h$

$$Q = \mathrm{th}^2\,(\beta h), \quad \lambda = 0$$

The resulting free energy is then calculated from (22) and (25), to be

$$\frac{\overline{F}}{N} = -\frac{1}{4T} - T \ln 2 - T \ln \mathrm{ch}\, \frac{h}{T}$$

This replica-symmetric solution is indeed stable for large $T$ (we shall derive the precise phase diagram below) and reproduces correctly the value of the thermodynamic quantities in the high-temperature phase of the random energy model [8] This phase contains a single pure state $\bar{P}(q) = \delta(q - \mathrm{th}^2\,(\beta h))$, whose self-overlap is the square of the magnetization.

$$\frac{\overline{F}}{N} = -\frac{1}{4T} - T\ln 2 - T\ln \mathrm{ch}\frac{h}{T} \quad \Rightarrow \quad S = \ln 2 - \frac{1}{4T^2} + \ln \mathrm{ch}\frac{h}{T} - \frac{h}{T}\mathrm{th}\frac{h}{T},$$

The entropy in this phase

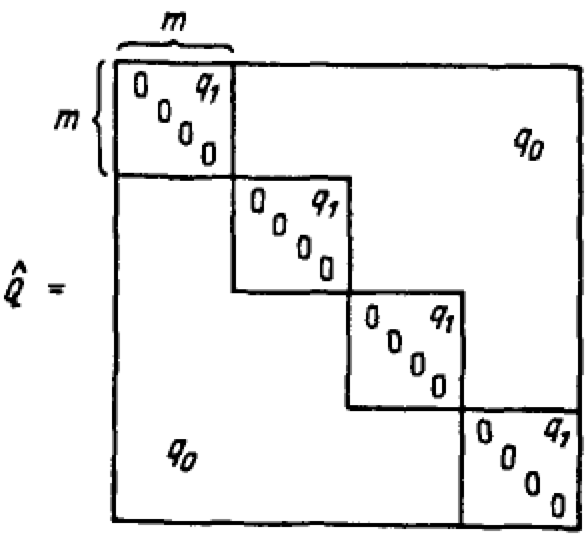
clearly becomes negative for $T \leqslant T_1(h)$, and therefore there must be a phase transition

Unlike the case in the $p=2$ model, the $q=\mathrm{th}^2(\beta h)$ solution is the only replica-symmetric one at all temperature (for $p \to \infty$) In fact, an analysis of the stability of this solution within the complete replica space (à la De Almeida–Thouless [11]) shows that it is always locally stable in zero field, as soon as $p>2$ In this respect,

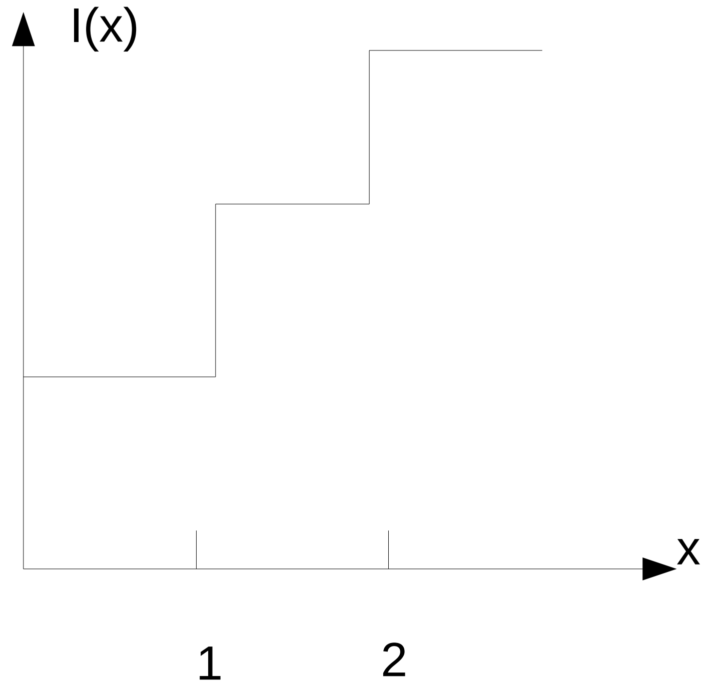
the $p=2$ SK model is somewhat special The spin glass transition must be (for $p>2$) a first-order one, at least as far as the order parameter function $q(x)$ is concerned In fact we shall show, in the $p \to \infty$ case, that the Edwards–Anderson order parameter, $q(1)$, jumps from 0 to 1 at $T_c$ However, since the order parameter is a function, and the discontinuity appears only on a set of zero measures, the transition turns out to be of second order in the thermodynamic sense.

# 1-step RSB

At the first step it is "natural" to divide all $n$ replicas into $n/m$ groups with $m$ replicas in each one (until now it is assumed, of course, that both $m$ and $n/m$ are integers).




$$Q_{ab} = \begin{cases} q_1 & \text{if } I\left(\frac{a}{m}\right) = I\left(\frac{b}{m}\right) \\ q_0 & \text{if } I\left(\frac{a}{m}\right) \neq I\left(\frac{b}{m}\right) \end{cases}$$




Here J=1

Free energy is a function of $q_0$, $q_1$ $\lambda_0$ $\lambda_1$ and m

$$\frac{1}{n} G = \ln 2 - \tfrac{1}{4}\beta^2(mq_0^p + (1-m)q_1^p) + \tfrac{1}{2}(m\lambda_0 q_0 + (1-m)\lambda_1 q_1)$$

$$-\tfrac{1}{2}\lambda_1 + \frac{1}{m}\int \mathrm{D}z_0 \ln \int \mathrm{D}z_1 \,\mathrm{ch}^m(z_0\sqrt{\lambda_0} + z_1\sqrt{\lambda_1 - \lambda_0} + \beta h)$$

For $p \to \infty$ the saddle-point equations are easy to solve First $\partial G/\partial q_i = 0$ implies

$$\lambda_i = \tfrac{1}{2}\beta^2 p q_i^{p-1}$$

For non-trivial symmetry breaking we must have $q_0 < q_1 \leqslant 1$, thus $\lambda_0 = 0$ If $q_1$ is also $<1$ then $\lambda_1 = 0$, in which case we will recover the symmetric solution $q_0 = q_1 = \mathrm{th}^2(\beta h)$. Hence $q_1 = 1$ and $\lambda_1 \sim \infty$

In this circumstance the double integral in $G$ is easily calculated, and we obtain $(\lambda_1 \sim \infty,\ \lambda_0 \sim 0)$

$$\frac{1}{n}G = -\tfrac{1}{4}\beta^2(mq_0^p + (1-m)q_1^p) + \tfrac{1}{2}(m\lambda_0 q_0 + (1-m)\lambda_1 q_1) - \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}m\lambda_1$$

$$+\frac{1}{m}\ln(2\,\mathrm{ch}\,(m\beta h)) - \tfrac{1}{2}m\lambda_0\,\mathrm{th}^2\,(m\beta h) + \mathrm{O}\,(\lambda_0^2, 1/\lambda_1)$$

Differentiating with respect to $\lambda_i$ then yields

$$q_0 = \mathrm{th}^2\,(\beta m h)\,, \qquad q_1 = 1$$

Finally the variation with respect to $m$ gives

$$m^2\beta^2 = 4\,[\ln 2 + \ln\mathrm{ch}\,(m\beta h) - m\beta h\,\mathrm{th}\,(m\beta h)]$$

This equation tells us that $m\beta = \beta_c$ is independent of the temperature, and $\beta_c$ is

$$\beta_c^2 = 4[\ln(2\,\mathrm{ch}\,(\beta_c h)) - \beta_c h\,\mathrm{th}\,(\beta_c h)]$$

Since $m \leqslant 1$, the solution exists only for $T < T_c = 1/\beta_c$

$T_c$ is precisely the value of the temperature $T_1(h)$, at which the entropy, eq (28b), of the high-temperature solution turns negative, and coincides with the critical line of the random energy model The free energy obtained for $T < T_c(H)$ can easily be calculated, using the above solution.

$$\frac{\overline{F}}{N} = -\frac{1}{2T_c} - h \operatorname{th} \frac{h}{T_c}$$

precisely the result found by Derrida [8], for the low-temperature phase of the random energy model The magnetization is given by $m = \operatorname{th}(h/T_c)$ and the magnetic susceptibility is temperature-independent ($\chi = 1/T_c \operatorname{ch}^2(h/T_c)$), as is also true in the SK model [6].

Here *the first breaking of replica symmetry gives the exact answer.*

(Appendix A)

$$q(x) = \operatorname{th}^2(\beta_c h)\theta(T/T_c - x) + \theta(x - T/T_c).$$

$$\overline{P(q)} = (T/T_c)\,\delta(q - \operatorname{th}^2(\beta_c h)) + (1 - T/T_c)\delta(q-1)$$

# General p > 2 spin glass

$$\langle Z^n \rangle = \operatorname*{Tr}_{\{S_i^\alpha\}_{\alpha=1,\ldots,n}} \exp\left[ \frac{p!}{4N^{p-1}T^2} \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} \left( \sum_{\alpha=1}^{n} S_{i_1}^\alpha S_{i_2}^\alpha, \ldots, S_{i_p}^\alpha \right)^2 + \beta h \sum_{i=1}^{N} \sum_{\alpha=1}^{n} S_i^\alpha \right]$$

where $\alpha$ is the replica index. Using the relation

$$1 = \frac{N}{2\pi} \int \prod_{\alpha \neq \beta} \mathrm{d}\lambda_{\alpha\beta} \mathrm{d}q_{\alpha\beta} \exp\left[ - \sum_{\alpha \neq \beta} \lambda_{\alpha\beta} \left( N q_{\alpha\beta} - \sum_{i=1}^{N} S_i^\alpha S_i^\beta \right) \right]$$

and taking the saddle point of the $\lambda$ and $q$ integrals, one has

$$\beta F = \lim_{n \to 0} \max \frac{1}{n} G(q_{\alpha\beta}, \lambda_{\alpha\beta}) - \tfrac{1}{4}\beta^2,$$

$$G(q_{\alpha\beta}, \lambda_{\alpha\beta}) = -\tfrac{1}{4}\beta^2 \sum_{\alpha \neq \beta} q_{\alpha\beta}^p + \tfrac{1}{2} \sum_{\alpha \neq \beta} \lambda_{\alpha\beta} q_{\alpha\beta} - \ln \operatorname*{Tr}_{\{S^\alpha\}_{\alpha=1,\ldots,n}} \exp\left( \tfrac{1}{2} \sum_{\alpha \neq \beta} \lambda_{\alpha\beta} S^\alpha S^\beta + \beta h \sum_{\alpha} S^\alpha \right).$$

## Replica symmetric solutions $\quad q_{\alpha\beta} = q$ and $\lambda_{\alpha\beta} = \lambda$

$$\lim_{n\to 0} \frac{G}{n} = \tfrac{1}{4}\beta^2 q^p - \tfrac{1}{2}\lambda(q-1) - \int \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \ln 2\cosh(\sqrt{\lambda}\, x + \beta h).$$

The mean field equations for $\lambda$ and $q$ are then,

$$\lambda = \tfrac{1}{2}\beta^2 p q^{p-1}, \qquad q = \int \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \tanh^2(\sqrt{\lambda}\, x + \beta h) \qquad (11)$$

Eqs. (11) will now be considered for general values of $p$. For simplicity, the magnetic field $h$ will be set equal to zero. For all values of $p$, the equations have the solution $\lambda = q = 0$. A replica symmetric solution is stable provided that [10]

$$\frac{1}{p-1}\frac{q}{\lambda} - \int \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \operatorname{sech}^4(\sqrt{\lambda}\, x) > 0.$$

Since the first equation in (11) implies that $q/\lambda$ is proportional to $q^{2-p}$, the first term is infinite for $p > 2$ and so the solution is stable at all temperatures. This is in contrast to the case $p = 2$ where the solution is stable only for $T > 1$.

There are other solutions to eqs. (11) when $p$ is finite. These solutions always involve a jump in the order parameters $q$ and $\lambda$. They can be found explicitly in expansions around the limits $p \to \infty$ and $p \to 2$.

These solutions are all unstable (can be checked directly for p=2+ε and p >> 1)

The only stable replica symmetric solution in zero field is then $q = 0$. However, it is not satisfactory at low temperatures. For any $p$, the entropy per site $s = \ln 2 - 1/4T^2$ derived from the free energy of this solution (3) becomes negative below a temperature $T = 1/2\sqrt{\ln 2}$. There must then be a phase transition at a temperature greater than this which involves replica symmetry breaking. Since the high-temperature solution does not become unstable, the new solution cannot be close to it and so there must be a jump in the order parameter function at the critical temperature.

# Solutions with one replica breaking

Large p first

$n$ replicas are grouped into $n/x_0$ clusters of $x_0$ replicas.

$$\lim_{n\to 0}\frac{G}{n} = \tfrac{1}{4}\beta^2\left(x_0 q_0^p + (1-x_0)q_1^p\right) - \tfrac{1}{2}\left(x_0\lambda_0 q_0 + (1-x_0)\lambda_1 q_1\right)$$

$$+ \tfrac{1}{2}\lambda_1 - \frac{1}{x_0}\int\frac{\mathrm{d}z_0}{\sqrt{2\pi}}\,\mathrm{e}^{-z_0^2/2}\ln\int\frac{\mathrm{d}z_1}{\sqrt{2\pi}}\,2^{x_0}\cosh^{x_0}\left(z_0\sqrt{\lambda_0} + z_1\sqrt{\lambda_1-\lambda_0} + \beta h\right)$$

The expansion around the large-$p$ limit will turn out to be an expansion in $2^{-p}$ and so should be rapidly convergent. For simplicity only the case of $h = 0$ will be considered. Assuming that at large $p$, $q_0$ is small, $q_1$ is close to 1 and that $\lambda_1 x_0^2$ is large, then,

$$\frac{G}{n} = x_0\left(\tfrac{1}{4}\beta^2 q_0^p - \tfrac{1}{2}\lambda_0 q_0\right) + (1-x_0)\left(\tfrac{1}{4}\beta^2 q_1^p + \tfrac{1}{2}\lambda_1(1-q_1)\right) - \frac{\ln 2}{x_0} + \frac{\xi(x_0)}{x_0}\,\frac{\mathrm{e}^{-x_0^2\lambda_{1/2}}}{\sqrt{\lambda_1}}$$

where

$$\xi(x_0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}z\,\left(2\cosh(x_0 z) - 2^{x_0}\cosh^{x_0} z\right).$$

Differentiation with respect to $q_i$, $\lambda_i$ and $x_0$ gives mean field equations whose solution is

$$q_0 = 0, \qquad q_1 = 1 - \frac{x_0 \xi(x_0)}{1 - x_0} \frac{e^{-x_0^2 p \beta^2/4}}{\sqrt{\frac{1}{2} p \beta^2}} \qquad \frac{1}{4}\beta^2 x_0^2 = \ln 2 + x_0^2 \frac{d}{dx_0} \frac{\xi(x_0)}{x_0} \frac{e^{-x_0^2 p \beta^2/4}}{\sqrt{\frac{1}{2} p \beta^2}} \tag{21}$$

The free energy of this solution coincides with the free energy of the high-temperature replica symmetric solution, $q = 0$ only at $x_0 = 1$. Substitution of $x_0 = 1$ into eq. (21) then gives the critical temperature,

$$T_c = \frac{1}{2\sqrt{\ln 2}} \left( 1 + 2^{-(p+1)} \sqrt{\frac{\pi}{p (\ln 2)^3}} \right)$$

The break point $x_0$ is given by,

$$x_0 = \frac{T}{T_c} \left( 1 - 2^{-(p+1)} \frac{T}{T_c} \xi\left(\frac{T}{T_c}\right) \sqrt{\frac{2p}{\ln 2}} \right)$$

$\xi(1) = 0$

Now we consider region of p slightly larger than 2:

$p = 2 + \varepsilon$, where $\varepsilon$ is small,

the jump near $T_c$ is small

a solution, $\lambda_0 = q_0 = 0$ and $q_1$, $\lambda_1$ and $x_0$ at $h = 0$ are obtained from maximisation of

$$\frac{G}{n} = (1 - x_0)\left\{ \tfrac{1}{4}\beta^2 q_1^p - \tfrac{1}{2}\lambda_1 q_1 + \tfrac{1}{4}\lambda_1^2 - \tfrac{1}{6}\lambda_1^3(2 - x_0) + \tfrac{1}{24}\lambda_1^4\left(3x_0^2 - 15x_0 + 17\right)\right\}$$

The solution, for $q_1$ and $x_0$

$$q_1 = \frac{3}{2}\frac{\varepsilon}{x_0} + o(\varepsilon^2) \qquad (26)$$

$$\tfrac{1}{2}\beta^2 p\left(\frac{3\varepsilon}{2x_0}\right)^{\varepsilon} = 1 + \frac{3}{2}\frac{\varepsilon(2 - x_0)}{x_0} \qquad (27)$$

The critical temperature is again at $x_0 = 1$ and is given by

$$T_c = \left(\tfrac{3}{2}\varepsilon\right)^{\varepsilon/2}\left(1 - \tfrac{1}{2}\varepsilon\right).$$

The new phase is stable for a finite range of temperatures. In this phase, the system chooses the valleys with the lowest free energy. In the thermodynamic limit these valleys have overlap zero with probability 1. For $p \to \infty$, the free-energy valleys consist of single configurations whose energies are independent gaussian random variables and the system freezes completely into its ground state at $T_c$. For large but finite $p$, there are small correlations between the levels and the freezing is incomplete. Since the size of the valleys is small, the self-overlap is close to 1. As $p$ decreases, the size of valleys near $T_c$ increases and the transition becomes continuous as $p \to 2$. Since the self-overlap of valleys is non-zero, each valley behaves qualitatively as though it is in a magnetic field which tends to $\infty$ as $p \to \infty$ and to zero as $p \to 2$. In the following sections, it will be shown that there is an explicit relation between $p$ and the magnetic field of the Sherrington-Kirkpatrick model in expansions around $p = 2$ and around $p = \infty$ near the instability temperature of the phase.

# Stability of the first low-temperature phase

Stability condition

In the space where $\alpha$ and $\beta$ belong to the same group.

$$\frac{1}{p-1}\frac{q_1}{\lambda_1}-\frac{\int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \cosh^{x_0-4}\sqrt{\lambda_1}\, z_1}{\int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \cosh^{x_0}\sqrt{\lambda_1}\, z_1} > 0 .$$

For large $p$, the phase is stable provided $T > T_2$ where $T_2 = \frac{2}{3}\frac{p^{3/2}}{\sqrt{\pi}} 2^{-p}$

For $p = 2 + \varepsilon$, the solution (26) is stable. close to $T_c$

However, the expansion leading to eqs.

(26) and (27) is not correct when $x_0$ is small.

In this case, the solution to the mean field equations derived from (25) is

$$\lambda_1 = \tfrac{1}{2} c\sqrt{\varepsilon}\ , \qquad x_0 = \frac{6 + c^2}{2c}\sqrt{\varepsilon}$$

Where c is related to the temperature via

$$T = \sqrt{\tfrac{1}{2} p}\, \varepsilon^{-\varepsilon/4}\left(1 - \tfrac{1}{2} c\sqrt{\varepsilon} + o(\varepsilon)\right)$$

it is stable provided $c < \sqrt{6}$ and so $T_2$ is given by

$$T_2 = \sqrt{\tfrac{1}{2} p}\, \varepsilon^{-\varepsilon/4}\left(1 - \sqrt{6\varepsilon}\,\right); \qquad T_c = \left(\tfrac{3}{2}\varepsilon\right)^{\varepsilon/2}\left(1 - \tfrac{1}{2}\varepsilon\right).$$

$$T_c - T_2 \approx \sqrt{6\varepsilon}$$

Second transition temperatures $T_2$ are the same as for the SK model in presence of external field h

$$h = \sqrt{2p \ln 2} + o(\ln p), \qquad \text{as } p \to \infty$$

$$h = 6^{1/4}(p-2)^{3/4}\left(1 + o\left((p-2)^{1/2}\right)\right), \qquad \text{as } p \to 2 .$$

$T_2$ is the temperature of local instability.
Below it continuous RSB state is formed

# Random directed polymer problem

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XY model with transverse random field on a Bethe lattice

$$H_{XY} = -2\sum_i \xi_i s_i^z - \sum_{(ij)} M_{ij}(s_i^+ s_j^- + s_i^- s_j^+)$$

Static mean-field approximation

$$H_j^{cav\text{-MF}} = -\xi_j \sigma_j^z - \sigma_j^x \frac{g}{K}\sum_{k=1}^{K} \langle \sigma_k^x \rangle$$

This implies that $B_j = \frac{g}{K}\sum_{k=1}^{K}\langle \sigma_k^x \rangle$, giving the recursion equation relating the $B$ fields

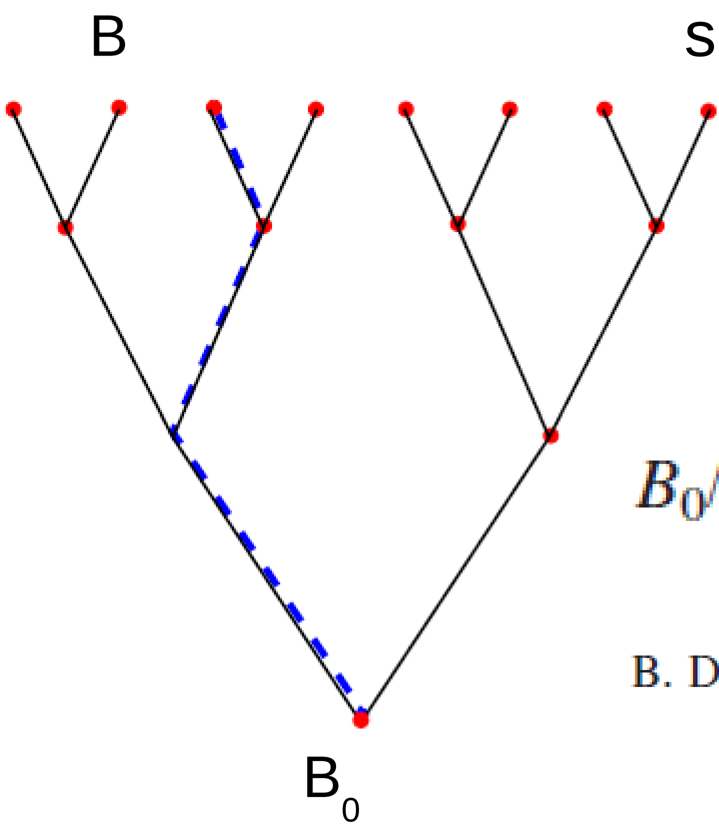
$$p(\xi) = \frac{1}{2}\theta(1-|\xi|)$$

$$B_j = \frac{g}{K}\sum_{k=1}^{K} \frac{B_k}{\sqrt{B_k^2 + \xi_k^2}} \tanh \beta\sqrt{B_k^2 + \xi_k^2}$$

$K \to \infty$ ⇒

$$1 = g\int d\xi p(\xi) \frac{\tanh(\beta\sqrt{\xi^2 + B^2})}{\sqrt{\xi^2 + B^2}}.$$

Search for instability leading to nonzero self-consistent B




$$B_0/B = \Xi \equiv \sum_P \prod_{k \in P} \left[ \frac{g}{K} \frac{\tanh(\beta \xi_k)}{\xi_k} \right]$$

B. Derrida and H. Spohn, J. Stat. Phys. **51**, 817 (1988).

Traveling wave method

where the sum is over all paths going from the root to the boundary and the product $\Pi_{n \in P}$ is over all edges along the path $P$. The response $\Xi$ is nothing but the partition function for a directed polymer (DP) on a tree, where the energy of each edge is $e^{-E_k} = (g/K)[\tanh(\beta\xi_k)/\xi_k]$ and the temperature has been set equal to one.

# Replica method

The DP partition function $\Xi$, defined in Eq. (9), depends on the random quenched variables $\xi_n$. One expects that the free-energy in a short-range-interaction problem is a self-averaging quantity, so the value of log $\Xi$ for a *typical* sample is obtained from the quenched average of log $\Xi$ over these random variables, denoted by $\overline{\ln \Xi}$. In the replica method one computes it by writing

$$\overline{\ln \Xi} = \lim_{n\to 0}(\overline{\Xi^n} - 1)/n \quad \Xi^n = \prod_{a=1,\ldots,n}\left[\sum_{P_a}\prod_{k\in P_a}\frac{g}{K}\frac{\tanh(\beta\xi_k)}{\xi_k}\right].$$

The average of $\Xi^n$ is obtained by a sum over $n$ paths

$$\overline{\Xi^n} = \sum_{P_1,\ldots,P_n}\prod_k \overline{\left[\frac{g}{K}\frac{\tanh(\beta\xi_k)}{\xi_k}\right]^{r_k}}$$

The RS solution assumes that the leading contribution to Eq. (12) comes from nonoverlapping independent paths ($r_k$ $=1$). This gives

$$\overline{\Xi^n} = K^{Ln}\left[\frac{g}{K}\int_{-1}^{1}\frac{d\xi}{\xi}\tanh(\beta\xi)\right]^{Ln} = \exp(Ln[\log(g/K)+f(1)]) = (\bar{\Xi})^n$$

The RSB solution assumes that the leading contribution to Eq. (12) comes from patterns of $n$ paths which consist of $n/x$ groups of $x$ identical paths, where the various groups go through distinct edges. This gives

$$\overline{\log\,\Xi} = \frac{L}{x}\ln K\overline{\left[\frac{g}{K}\frac{\tanh(\beta\xi_k)}{\xi_k}\right]^x} \equiv L\left[\ln\left(\frac{g}{K}\right)+f(x)\right]$$

for $\beta<\beta_{\mathrm{RSB}}$,

$f(x)$ is minimal at the boundary, $x=1$

where

$$f(x) = \frac{1}{x}\ln\left\{K\int_{-1}^{1}\frac{d\xi}{2}\left[\frac{\tanh(\beta\xi)}{\xi}\right]^x\right\}$$

$\beta>\beta_{\mathrm{RSB}}$, the function $f(x)$ has a minimum inside the interval (0,1) at some value $x=m<1$, this corresponds to the spontaneous breakdown of the replica symmetry in the DP problem.

$$\overline{\Xi^n} = K^{Ln}\left[\frac{g}{K}\int_{-1}^{1}\frac{d\xi}{\xi}\tanh(\beta\xi)\right]^{Ln} = \exp(Ln[\log(g/K) + f(1)]) = (\bar{\Xi})^n$$

The RSB solution assumes that the leading contribution to Eq. (12) comes from patterns of $n$ paths which consist of $n/x$ groups of $x$ identical paths, where the various groups go through distinct edges. This gives

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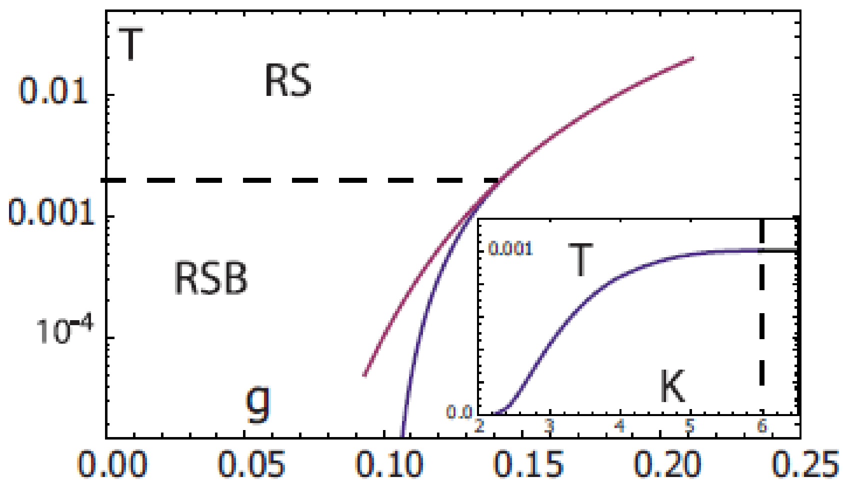
$\beta > \beta_{RSB}$, the function $f(x)$ has a minimum inside the interval (0,1) at some value $x = m < 1$, this corresponds to the spontaneous breakdown of the replica symmetry in the DP problem.

These two regimes of the DP problem are qualitatively very different. In the RS regime the measure on paths defined in Eq. (9) is more or less evenly distributed among all paths. On the contrary, the RSB regime is a glass phase where the measure condenses onto a small number of paths. An order parameter which distinguishes between these two phases is the participation ratio $Y=\Sigma_P w_P^2$, where $w_P$ is the relative weight of path $P$ in the measure in Eq. (9). It is easy to see that $Y=0$ in the RS phase. In the RSB phase, the value of $Y$ is finite and non self-averaging (it depends on the realization of the $\xi$'s), and its average is given by $1-m$. This glass transition, and the nature of the RSB glass phase, are identical to the ones found in the random energy model.[38,39]




At zero temperature

$$f(x) = \frac{1}{x}\ln\left(\frac{K}{1-x}\right)$$

$$g_c e^{1/(e g_c)} = K,$$

$$m = 1 - e g_c.$$

Physical meaning of the RSB in the present problem: broad distribution of local fields B without even $1^{st}$ momentum <B>:

$$P(B) = \frac{B_0^m}{B^{1+m}}. \qquad \text{at large } B \geqslant B_0$$

Typical value of B

This power-law tail at large $B$ translates into the behavior of the Laplace transform

$$\mathcal{P}(s) \approx 1 - (sB_0)^m \quad \text{at} \quad sB_0 \leqslant 1.$$

# Few problems to solve

- Calculate critical behavior of entropy at the glass transition in p-spin glass model, in the limits $p \gg 1$ and $p-2 \ll 1$
- Calculate average participation ratio $\langle Y \rangle$ and its square $\langle Y^2 \rangle$ in the RSB phase of the random directed polymer model
- Find critical value of $K(g)$ for the RDPM described in the lecture, such that temperature-driven transition to the ordered state must be described via RSB scheme.