Parisi Scheme for ReplicaSymmetryBreaking

Useful review: **Physics of the spin-glass state**

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Fundamental book: M. Mezard, G. Parisi and M. Virasoro

Spin Glass Theory and Beyond

1. 1-step RSB for the SK model

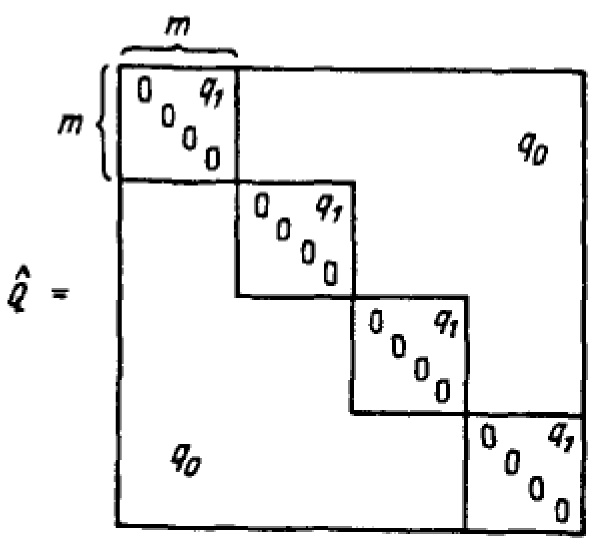
2. Hierarchical Parisi scheme

3. Parisi function q(x) and its calculation near T_c

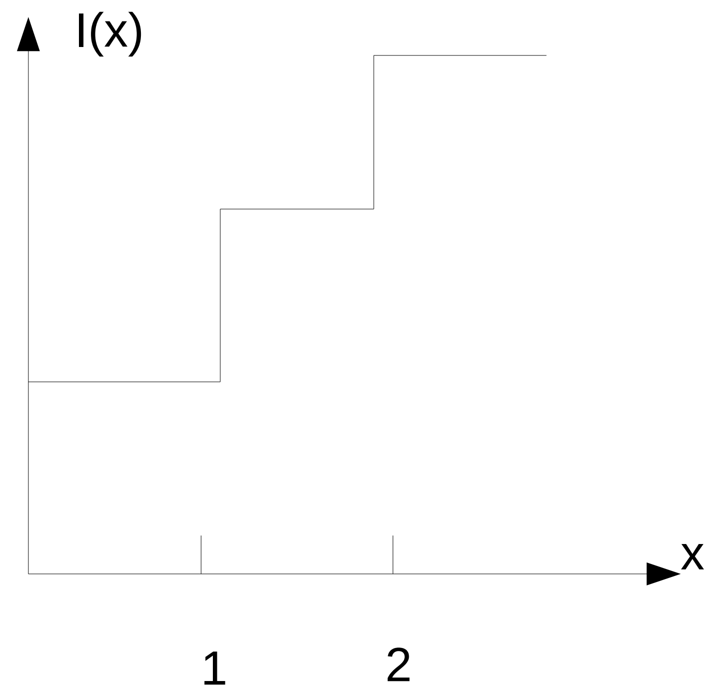
4. Interpretation: overlap distribution P(q) and ultrametricity

1-step RSB

At the first step it is "natural" to divide all n replicas into n/m groups with m replicas in each one (until now it is assumed, of course, that both m and n/m are integers).



$$Q_{ab} = \begin{cases} q_1 & \text{if } I\left(\frac{a}{m}\right) = I\left(\frac{b}{m}\right) \\ q_0 & \text{if } I\left(\frac{a}{m}\right) \neq I\left(\frac{b}{m}\right) \end{cases}$$



$$f[\hat{Q}] = -\tfrac{1}{4}\beta + \frac{\beta}{2n} \sum_{a<b}^{n} Q_{ab}^2 - \frac{1}{\beta n} \log Z([\hat{Q}])$$

$$Z([\hat{Q}]) = \sum_{\sigma_a} \exp\left(\beta^2 \sum_{a<b}^{n} Q_{ab}\sigma_a\sigma_b\right).$$

Here J=1

$$\sum_{a<b}^{n} Q_{ab}\sigma_a\sigma_b=\frac{1}{2}\left[q_0\left(\sum_{a=1}^{n}\sigma_a\right)^2+(q_1-q_1)\times\sum_{k=1}^{n/m}\left(\sum_{c_k=1}^{m}\sigma_{c_k}\right)^2-nq_1\right]$$

Here k numbers the replica blocks and c_k numbers the replicas inside the blocks. After the Gaussian transformation in $Z[\hat{Q}]$ for each of the squares in the above equation,

$$Z[q_1,q_0,m]=\int\frac{dz}{\sqrt{2\pi q_0}}\exp\left(-\frac{z^2}{2q_0}\right)\prod_{k=0}^{n/m}\left(\int\frac{dy_k}{\sqrt{2\pi(q_1-q_0)}}\langle\exp\left(-\frac{y_k^2}{2(q_1-q_0)}\right)\right.$$

$$\sum_{\sigma_a}\exp\left\{\beta\left[z\sum_{a}^{n}\sigma_a+\sum_{k=0}^{n/m}y_k\left(\sum_{c_k=1}^{m}\sigma_{c_k}\right)\right]-\frac{1}{2}\beta^2nq_1\right\}.$$

The summation over the spins gives:

$$Z[q_1,q_0,m]=\int\frac{dz}{\sqrt{2\pi q_0}}\exp\left(-\frac{z^2}{2q_0}\right)\left[\int\frac{dy}{\sqrt{2\pi(q_1-q_0)}}\exp\left[-\frac{y^2}{2(q_1-q_0)}\right]\right.$$

$$\left.\times\left\{2\cosh\beta(z+y)\right\}^m\right]^{n/m}\exp\left(\frac{1}{2}\beta^2nq_1\right).$$

$$f[\hat{Q}] = -\tfrac{1}{4}\beta + \frac{\beta}{2n} \sum_{a<b}^{n} Q_{ab}^2 - \frac{1}{\beta n} \log Z([\hat{Q}])$$

Second term:

$$\frac{\beta}{2n} \sum_{a<b}^{n} Q_{ab}^2 = \frac{\beta}{4n} \left[q_1^2 m(m-1) \frac{n}{m} + q_0^2 \left(n^2 - m^2 \frac{n}{m} \right) \right]$$

$$= \frac{\beta}{4} [q_1^2 (m-1) + q_0^2 (n-m)].$$

Now the limit $n \to 0$ has to be taken.

formal analytic continuation turns $1 \leqslant m \leqslant n$. into $0 \leqslant m \leqslant 1$

$$f(q_1, q_0, m) = -\tfrac{1}{4}\beta(1 + m q_0^2 + (1-m) q_1^2 - 2q_1) - \frac{1}{m} \int \frac{dz}{\sqrt{2\pi q_0}} \exp\left(-\frac{z^2}{2q_0} \right)$$

$$\ln\left[\int \frac{dy}{\sqrt{2\pi(q_1 - q_0)}} \exp\left(-\frac{y^2}{2(q_1 - q_0)} \right) (\cosh \beta(z+y))^m \right] - \ln 2$$

in the cases $m=0$ and $m=1$ replica symmetric solution is recovered with $q=q_0$ and $q=q_1$ respectively.

Minimum or maximum ?

Note now another essential point. Actually, in the replica formalism one is looking for the maxima and not for the minima of the free energy. The formal reason is that in the limit $n \to 0$ the number of components of the order parameter $\hat{Q}$ becomes negative. For example, in the case of the one-step RSB each line of the matrix $\hat{Q}$ contains $(m-1) < 0$ components which are equal to q_1, and $(n-m) \to$

$-m < 0$ components which are equal to q_0.

At high temperatures it is sufficient to consider the quadratic term only

$$\lim_{n \to 0} \left[\frac{\beta}{n} \sum_{a<b} Q_{ab}^2 \right] = -\frac{\beta}{2} [(1-m) q_1^2 + m q_0^2].$$

It is obvious that for $0 \leqslant m \leqslant 1$ the "correct extremum" in which the Hessian is positive, is the maximum and not the minimum with respect to q_0 and q_1.

1-step RSB saddle-point

Extremal of the free energy $f(m,q_0,q_1)$ over all 3 variables

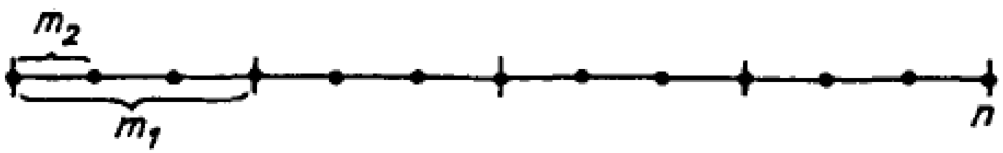
The result of numerical solution is

1) In the low-temperature phase $T<1$ the function f has indeed a maximum at a certain point: $0 \leqslant m(T) \leqslant 1$; $0 \leqslant q_0(T) \leqslant 1$; $0 \leqslant q_1(T) \leqslant 1$ (both for $T \to 1$ and $T \to 0$ one gets $m(T) \to 0$).

2) Although the entropy at low temperatures still becomes negative, this negative value appears to be much smaller than that of the replica-symmetric solution: $S(T=0) \simeq -0.01$ (while for the RS solution $S(T=0) \simeq -0.17$)

3) The most negative eigenvalue of the Hessian near T_c is equal to $-c(T-T_c)^2/9$ (c is some positive number), while for the RS solution it is equal to $-c(T-T_c)^2$. So that, in a sense, the instability is reduced by a factor 9.

Full-scale RSB: Parisi scheme



Parisi, G. (1979) *Phys. Rev. Lett.* **43,** 1754.
Parisi, G. (1980). *J. Phys.* **A13,** 1101, 1887, L115.
Parisi, G. (1980) *Phil. Mag.* **B41,** 677.
Parisi, G. (1980). *Phys. Rep.* 67, 97.

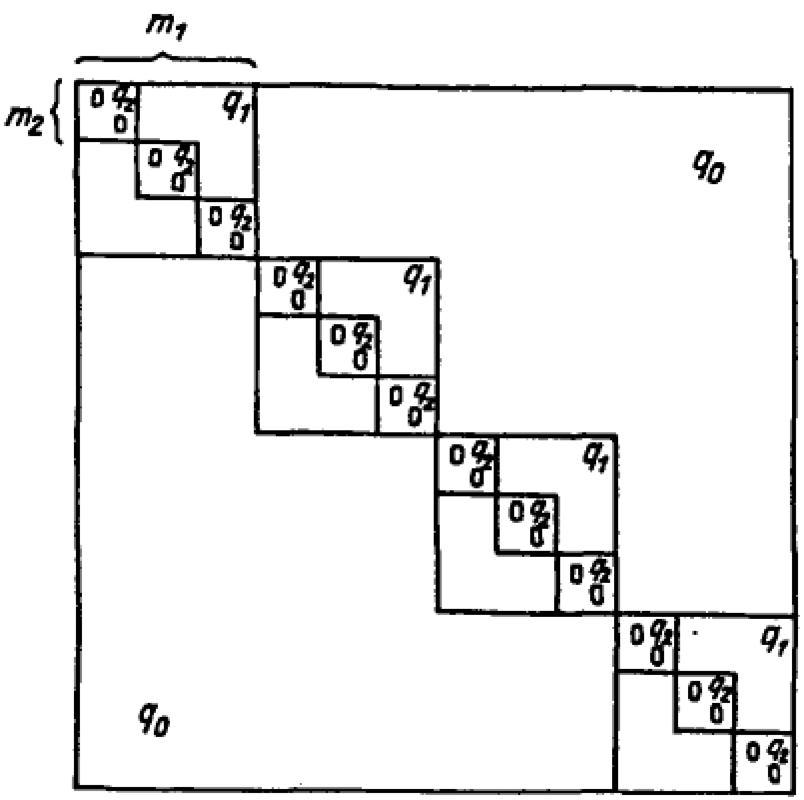


FIG. 15. The explicit form of the matrix Q_{ab} at the two-step r symmetry breaking.

m_i $(i=1,2,...,k+1)$ such that $m_0=n$, $m_{k+1}=1$ and all m_i/m_{i+1} would be integers. Next, let us divide n replicas into n/m_1 groups such that each group would consist of m_1 replicas; divide each group of m_1 replicas into m_1/m_2 sub-groups so that each group would consist of m_2 replicas;

$$Q_{ab}=q_i \quad \text{where} \quad I\left(\frac{a}{m_{i+1}}\right)=I\left(\frac{b}{m_{i+1}}\right)$$

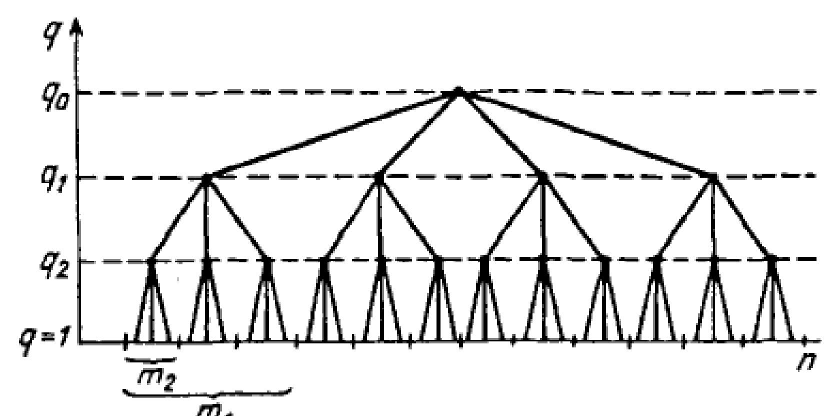


FIG. 14. The definition scheme for the matrix elements Q_{ab} at the two-step replica symmetry breaking.

Free energy is a function of k+1 variables q_k and k variables m_k

$$
\begin{aligned}
&f[q_0,q_1,\ldots,q_k;m_1,m_2,\ldots,m_k] \\
&= -\frac{1}{4}\beta\left[1+\sum_{i=1}^{k}(m_{i+1}-m_i)q_i^2-2q_k\right]-\frac{1}{m_1}\int\frac{dz_0}{\sqrt{2\pi q_0}} \\
&\times\exp\left(-\frac{z_0^2}{2q_0}\right)\ln\left\{\int\frac{dz_1}{\sqrt{2\pi(q_1-q_0)}}\right. \\
&\times\exp\left(-\frac{z_1^2}{2(q_1-q_0)}\right)\left[\int\frac{dz_2}{\sqrt{2\pi(q_2-q_1)}}\right. \\
&\times\exp\left(-\frac{z_2^2}{2(q_2-q_1)}\right)\Bigg]\ldots\left[\int\frac{dz_k}{\sqrt{2\pi(q_k-q_{k-1})}}\right. \\
&\times\exp\left(-\frac{z_k^2}{2(q_k-q_{k-1})}\right) \\
&\times\left.\left(2\cosh\beta\left(\sum_{i=0}^{k}z_k\right)\right)^{m_k}\right]^{m_{k-1}/m_k}\ldots\Bigg]^{m_2/m_3}\Bigg]^{m_1/m_2}\Bigg\}.
\end{aligned}
$$

To get exact solution, the limit $k\to\infty$ must be taken

It means we should consider order parameter function $q(x)$

General equations derived in G. Parisi, J. Phys. **A13**, L115 (1980).
B. Duplantier, J. Phys. **A14**, 283 (1981).

Free energy:

$$f[q(x)] = -\tfrac{1}{4}\beta\left[1 + \int_0^1 \mathrm{d}x q^2(x) - 2q(1)\right] - \frac{1}{\beta} A[q(x)]$$

where

$$A[q(x)] = \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{\sqrt{2\pi q(0)}} \exp\left(-\frac{z^2}{2q(0)}\right) g(0;\quad z+h)$$

and the function $g(x,y)$ is obtained from the following nonlinear differential equation:

$$\frac{\partial g(x;\ y)}{\partial x} = -\frac{1}{2}\frac{\mathrm{d}q(x)}{\mathrm{d}x}\left[\frac{\partial^2 g(x;\ y)}{\partial y^2} + x\left(\frac{\partial g(x;\ y)}{\partial y}\right)^2\right]$$

with the boundary condition:

$$g(1;\ y) = \ln[2\cosh(\beta y)].$$

One can prove that q(x) is monotonic function, thus its inverse x(q) exists

Then:

the saddle-point equations $\delta f/\delta q(x)=0$ Are equivalent to

$$q=\int dy\, m^2(q;\,y)$$

where the function $m(q;\,y)\equiv\partial g/\partial y$ is obtained from

$$\frac{\partial m(q;\,y)}{\partial q}=-\frac{1}{2}\left[\frac{\partial^2 m(q;\,y)}{\partial y^2}+2x(q)m(q;\,y)\frac{\partial m(q;\,y)}{\partial y}\right].$$

With boundary condition m(1,y) = tanh (β y)

There two equations constitute a system of functional equations which determine q(x) function and its inverse x(q)

Solution for q(x) slightly below T_c

$$f[\hat{Q}] = -\tfrac{1}{4}\beta + \frac{\beta}{2n} \sum_{a<b}^{n} Q_{ab}^2 \quad - \frac{1}{\beta n} \log\left[\sum_{\sigma_a} \exp\left(\beta^2 \sum_{a<b}^{n} Q_{ab}\sigma_a\sigma_b \right) \right]$$

Can be analyzed by expansion in power of Q_{ab}

The result of the expansion up to the fourth order is:

$$f[\hat{Q}] = \lim_{n\to 0} \frac{1}{n} \left[-\frac{1}{2}\tau \operatorname{Tr}(\hat{Q})^2 - \frac{1}{6}\operatorname{Tr}(\hat{Q})^3 - \frac{1}{12}\sum_{a,b} Q_{ab}^4 \quad + \frac{1}{4}\sum_{a,b,c} Q_{ab}^2 Q_{ac}^2 - \frac{1}{8}\operatorname{Tr}(\hat{Q})^4 \right]$$

$\tau = 1 - T \ll 1$

Term responsible for RSB

Higher-order terms
(to be checked later)

The above Eq. for f(Q) should be expressed via q(x) function

$$f[\hat{Q}] = \lim_{n\to 0} \frac{1}{n}\left[-\frac{1}{2}\tau\,\mathrm{Tr}(\hat{Q})^2 - \frac{1}{6}\mathrm{Tr}(\hat{Q})^3 - \frac{1}{12}\sum_{a,b} Q_{ab}^4\right.$$

$$\lim_{n\to 0}\frac{1}{n}\sum_{a,b}[Q_{ab}]^l = \sum_{i=0}^{k}(m_i - m_{i+1})q_i^l \to -\int_0^1 \mathrm{d}x\, q^l(x)$$

$$\lim_{n\to 0}\frac{1}{n}\mathrm{Tr}(\hat{Q})^3 = \int_0^1 \mathrm{d}x\left[xq^3(x) + 3q(x)\int_0^x \mathrm{d}y\, q^2(y)\right]$$

For the free energy one obtains:

$$f[q(x)] = \frac{1}{2}\int_0^1 \mathrm{d}x\left[\tau q^2(x) - \frac{1}{3}xq^3(x) - q(x)\int_0^x \mathrm{d}y\, q^2(y) + \frac{1}{6}q^4(x)\right]$$

Variation of this expression with respect to the function $q(x)$ gives the following saddle-point equation:

$$2\tau q(x) - xq^2(x) - 2q(x)\int_x^1 \mathrm{d}y\, q(y) - \int_0^x \mathrm{d}y\, q^2(y) + \tfrac{2}{3}q^3(x) = 0.$$

$$2\tau q(x) - xq^2(x) - 2q(x)\int_x^1 \mathrm{d}y q(y) - \int_0^x \mathrm{d}y q^2(y) + \tfrac{2}{3} q^3(x) = 0.$$

Take derivative wrt x and obtain

$$q'(x)\left[2\tau - 2xq(x) - 2\int_x^1 \mathrm{d}y q(y) + 2q^2(x)\right] = 0.$$

It is equivalent to the pair of equations

$$2\tau - 2xq(x) - 2\int_x^1 \mathrm{d}y q(y) + 2q^2(x) = 0$$

or

$$q'(x) = 0.$$ Cannot be true for all x but can be so for some x

$$q(x) = \begin{cases} q_0, & 0 \leqslant x \leqslant x_0 \\ \dfrac{1}{2}x, & x_0 \leqslant x \leqslant x_1 \\ q_1, & x_1 \leqslant x \leqslant 1 \end{cases}$$

where

$$x_1 = 2q_1; \quad x_0 = 2q_0$$

This solution must be substituted to the original equation

Substituting this solution into the original saddle-point equation at the points $x = x_0$ and $x = x_1$ one gets:

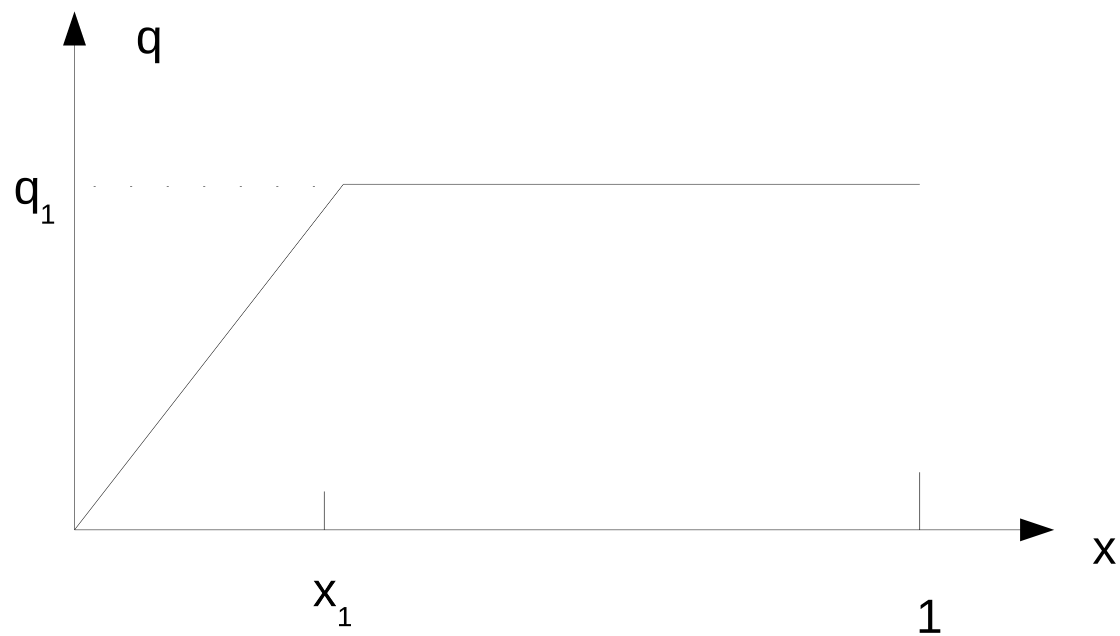
$$q_0[2\tau - 2q_1 + 2q_1^2] - \tfrac{4}{3} q_0^3 = 0,$$

$$q_1[2\tau - 2q_1 + 2q_1^2] - \tfrac{4}{3} q_0^3 = 0.$$

The solution of these equations is:

$$q_0 = 0,$$

$$q_1 = \tau + O(\tau^2).$$



Higher orders in τ

H.-J. Sommers **Parisi function $q(x)$ for spin glasses near T_c**

J. Physique Lett. **46** (1985) L-779 - L-785

$$q(1) = \tau + \tau^2 - \tau^3 + \frac{5}{2}\tau^4 - \frac{39}{2}\tau^5 + O(\tau^6),$$

$$x_1 = 2\tau - 4\tau^2 + 12\tau^3 - 69\tau^4 + O(\tau^5),$$

$$q(x) = \begin{cases} \frac{1}{2}(1 + 3\tau + 12\tau^3)x - \frac{1}{8}(1-\tau)x^3 + O(\tau^5) & (x \leqslant x_1), \\ q(1) & (x \geqslant x_1). \end{cases}$$

Stability of the Parisi solution

1. Parisi solution fulfills exactly the condition of marginal stability derived with the TAP equation approach (Bray & Moore)

$$1 = \beta^2 J^2 \overline{G^2(0)} = \beta^2 J^2 \overline{(1 - m^2)^2} = \beta^2 J^2 (1 - 2q + r)$$

Lowest nontrivial order:

$$(1- \tau)^2 = 1 - 2(\tau + \tau^2) + 3\,\tau^2$$

Problem to solve: check marginal stability in the next orders in τ

2. Stability check within replica scheme

2. Stability check within replica scheme (Dotsenko Feigelman, Ioffe)

Since the matrix $Q_{\alpha\beta}$ is not known explicitly and has a complicated structure, nobody has succeeded in rigorously proving the stability of the Parisi solution in general. However, it can be proved (in the leading-order approximation in τ) near the transition point T_c, where the matrix $K_{\alpha\beta\gamma\delta}$ can be expressed explicitly.

$$K_{\alpha\beta\gamma\delta} = -2(\tau + Q^2_{\alpha\beta})\delta_{\alpha\beta,\gamma\delta} - Q_{\alpha\gamma}\delta_{\beta\delta} - Q_{\alpha\delta}\delta_{\beta\gamma} - Q_{\beta\gamma}\delta_{\alpha\delta} - Q_{\beta\delta}\delta_{\alpha\gamma}.$$

Here only terms responsible for RSB are retained within 4-th order F(Q)

Straightforward but cumbersome calculations show that all eigenvalues of the $K_{\alpha\beta\gamma\delta}$ matrix are negative or zero, i.e. they have the same sign as the eigenvalues at high temperature; therefore the Parisi solution is stable. We restrict ourselves to a simpler problem of stability in the subspace that can be described by the Parisi ansatz. To study stability in this subspace, we can use the continuous description of the $n \to 0$ limit of the free energy (2.2.23) and expand it around the saddle point

to second order in small deviations $\tilde{q}(x)$ ($q = q^c + \tilde{q}$):

$$f = f_0\{q^c(x)\} + \frac{1}{2} T \int \mathrm{d}x \left[(\tau + q^2(x) - xq(x))\tilde{q}^2(x) - 2\tilde{q}(x) \int_0^x \mathrm{d}y\, \tilde{q}(y) q^c(y) - q(x) \int_0^x \mathrm{d}y\, q^2(y) \right].$$

The spectrum of the $K_{\alpha\beta\gamma\delta}$ matrix then follows from the equation

$$\delta f/\delta \tilde{q}(x) = \lambda q(x)$$

$$\lambda \tilde{q}'(x) = -2q'(x) \int_x^1 \mathrm{d}y\, \tilde{q}(y).$$

Conclusion: all eigenvalues are non-positive
(either negative, or =0)

Problem: to derive the above conclusion

In presence of magnetic field:

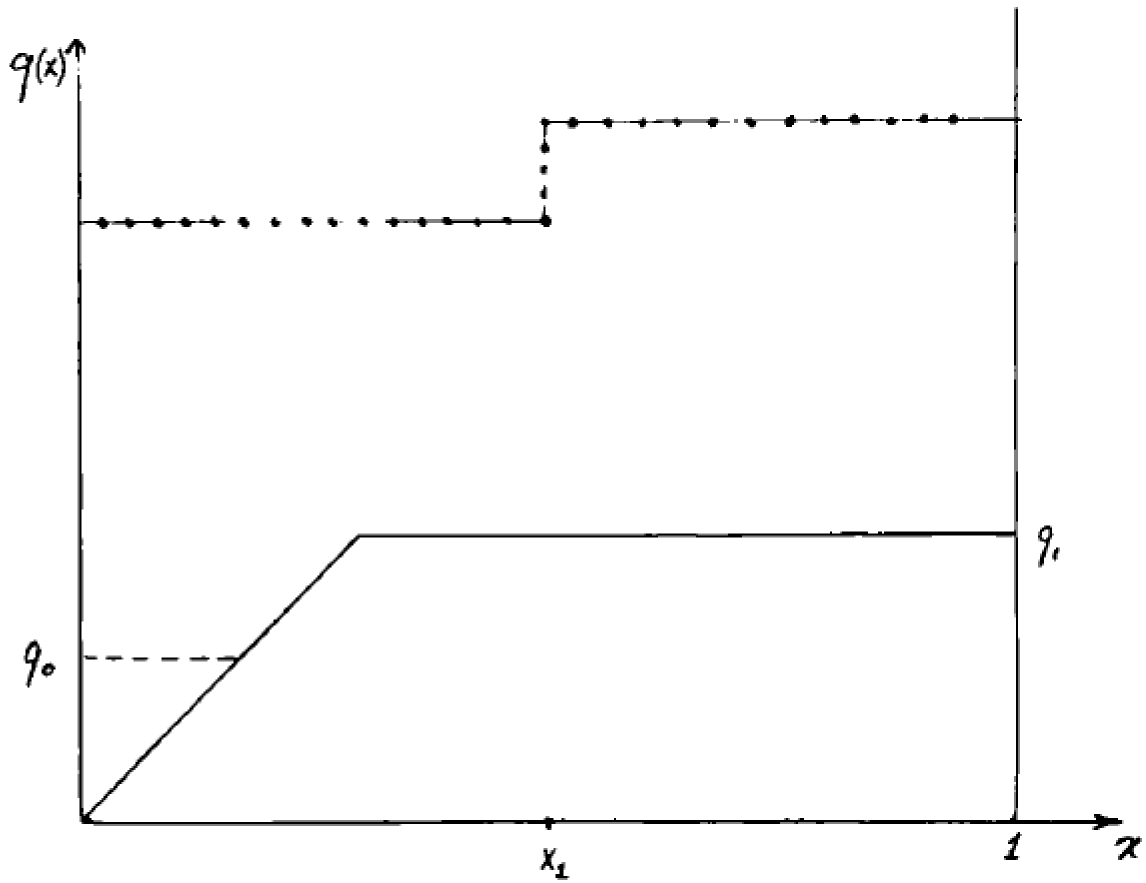


Figure 5 Solution $q(x)$ in the vicinity of T_c for zero (full line), small (dashed line) and large (dotted line) magnetic fields.

Low fields:

$$q(0) = \frac{3}{4}\left(\frac{h^2}{J^2}\right)^{2/3}$$

At the AT line q(0)=q(1)
RS solution

High fields:
See Sec.2.2
in the book by
Dotsenko,
Feigel'man &
Ioffe

De Almeida – Thouless line

J R L de Almeida†‡ and D J Thouless   J. Phys. A: Math. Gen., Vol. 11, No. 5, 1978. p983

Instability line we found right now coincides with the marginal stability condition

$(T/J)^2 = \langle (1- m_i^2)^2 \rangle$

If external field h>0 is present:

$$(T/J)^2 = \int \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2} \operatorname{sech}^4 \left(\frac{Jzq^{1/2} + h}{T} \right)$$

$$\tau_{AT} = \left(\frac{3}{4}\right)^{2/3} \left(\frac{h}{J}\right)^{2/3} \qquad (h \ll J)$$

$$T_{AT} = \frac{4}{3} \frac{J}{(2\pi)^{1/2}} \exp\left(-\frac{h^2}{2J^2}\right) \qquad (h \gg J).$$

Field-cooled susceptibility

$\chi_{FC} = (1 - q_{EA})/T$ where $q_{EA} = \overline{\langle \sigma_i \rangle^2} = \dfrac{\left[\sum_{\{\sigma\}} \sigma_i \exp(-H\{\sigma_i\}/T) \right]^2}{\left[\sum_{\{\sigma\}} \exp(-H\{\sigma\}/T) \right]^2}$

Should be derived !

$$= \lim_{n \to 0} \sum_{\{\sigma_i\}} \sigma_i^1 \sigma_i^2 \exp\left(- \frac{H\{\sigma_j^\alpha\}}{T} \right) = Q_{12} .$$

For the RSB case, symmetrization must de carried on

$$= - \frac{1}{n} \sum_{\alpha \neq \beta} Q_{\alpha\beta} .$$

$$\chi_{eq} = T^{-1}[1 - \int dx\, q(x)] = T_c^{-1}$$

Follows from q(o)=0

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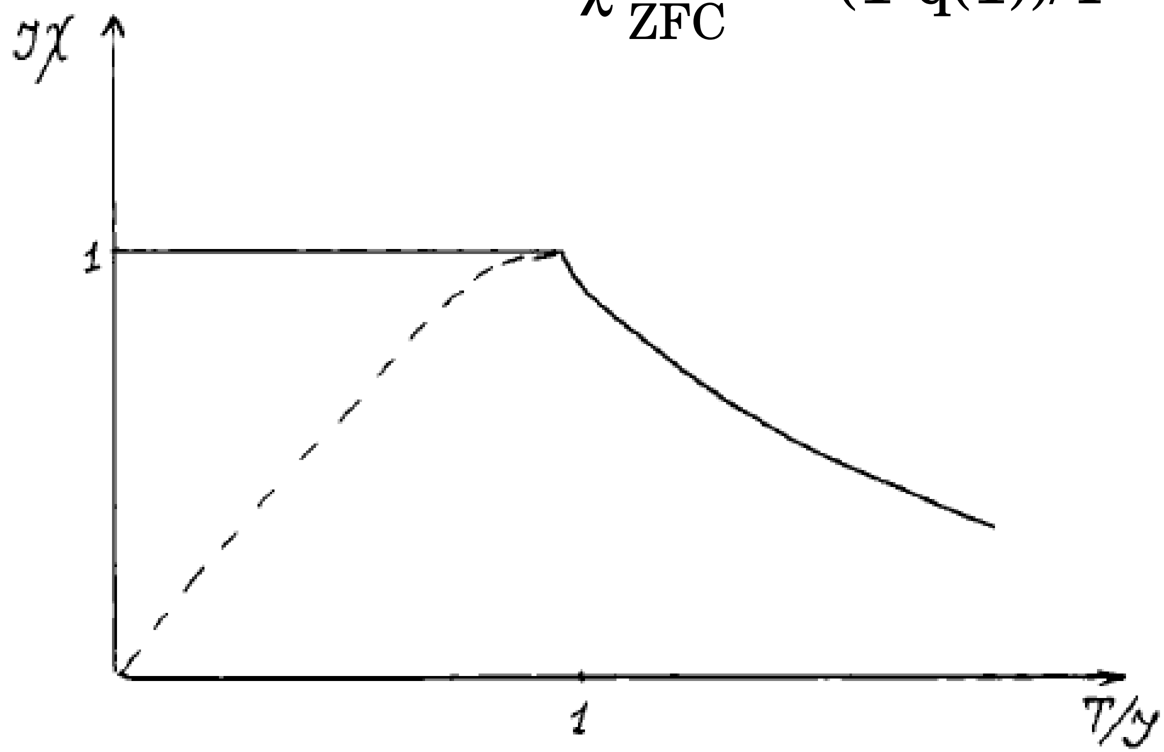
FC and ZFC Susceptibilities

$$\chi_{eq} = T^{-1}[1 - \int dx\, q(x)]$$

$$\chi_{ZFC} = (1-q(1))/T$$



Distribution of overlaps P(q)

Pure states: connected correlations decay to zero

$$\langle \sigma_i \rangle = m_i = \sum_\alpha w_\alpha m_i^\alpha .$$

$$\langle \sigma_i \sigma_j \rangle_c \equiv \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$$

Here the α's label the pure states and the w_α are their statistical weights $w_\alpha = \exp(-F_\alpha)$

the two-point correlation function can be represented as the linear combination

$$\langle \sigma_1 \sigma_2 \rangle = \sum_\alpha w_\alpha \langle \sigma_1 \sigma_2 \rangle_\alpha$$

The distance between the states α and β

$$d_{\alpha\beta} = \frac{1}{N} \sum_i^N (m_i^\alpha - m_i^\beta)^2$$

Complimentary quantity: overlap

$$q_{\alpha\beta} = \frac{1}{N} \sum_i^N m_i^\alpha m_i^\beta .$$

Probability distribution of overlaps: $$P_J(q)=\sum_{\alpha\beta} w_\alpha w_\beta \delta(q_{\alpha\beta}-q).$$

After averaging over disorder: $$P(q)=\langle\langle P_J(q)\rangle\rangle.$$

Consider the following series of correlation functions:

$$q_J^{(1)}=\frac{1}{N}\sum_i \langle\sigma_i\rangle^2 \qquad q_J^{(2)}=\frac{1}{N^2}\sum_{i_1 i_2}\langle\sigma_{i_1}\sigma_{i_2}\rangle^2 \qquad q_J^{(k)}=\frac{1}{N^k}\sum_{i_1\ldots i_k}\langle\sigma_{i_1}\ldots\sigma_{i_k}\rangle^2.$$

$$q_J^{(1)}=\frac{1}{N}\sum_i\left(\sum_\alpha w_\alpha\langle\sigma_i\rangle_\alpha\right)\left(\sum_\beta w_\beta\langle\sigma_i\rangle_\beta\right)$$

$$=\sum_{\alpha\beta} w_\alpha w_\beta q_{\alpha\beta}=\int \mathrm{d}q P_J(q)q;$$

$$q_J^{(2)}=\frac{1}{N^2}\sum_{i_1 i_2}\left(\sum_\alpha w_\alpha\langle\sigma_{i_1}\sigma_{i_2}\rangle_\alpha\right)\left(\sum_\beta w_\beta\langle\sigma_{i_1}\sigma_{i_2}\rangle_\beta\right)$$

$$=\sum_{\alpha\beta} w_\alpha w_\beta\left(\frac{1}{N}\sum_{i_1}\langle\sigma_{i_1}\rangle_\alpha\langle\sigma_{i_1}\rangle_\beta\right)\left(\frac{1}{N}\sum_{i_2}\langle\sigma_{i_2}\rangle_\alpha\langle\sigma_{i_2}\rangle_\beta\right)$$

$$=\sum_{\alpha\beta} w_\alpha w_\beta (q_{\alpha\beta})^2=\int \mathrm{d}q P_J(q)q^2;$$

$$q_J^{(k)}=\int \mathrm{d}q P_J(q)q^k.$$

The property was used:

$$\lim_{N\to\infty}\left[\frac{1}{N^2}\sum_{ij}(\langle\sigma_i\sigma_j\rangle_\alpha-\langle\sigma_i\rangle_\alpha\langle\sigma_j\rangle_\alpha)^2\right]=0.$$

Now we proceed to averaged description with replaces

$$q^{(1)} = \int \mathrm{d}q P(q) q \qquad q^{(k)} = \int \mathrm{d}q P(q) q^k$$

$$q^{(k)} = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_{a<b} [Q_{ab}]^k$$

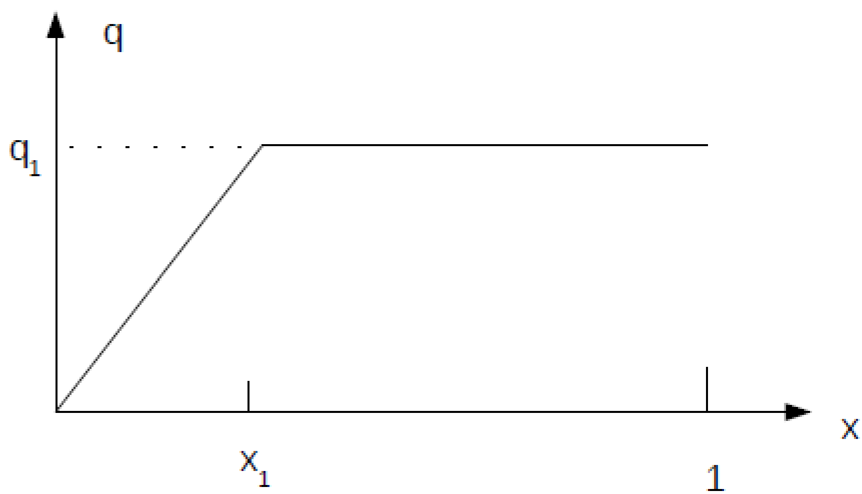
$$P(q) = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_{a<b} \delta(Q_{ab} - q).$$

$$\lim_{n\to 0} \frac{1}{n} \sum_{a,b} [Q_{ab}]^l = -\sum_{i=0}^{k} (m_{i+1} - m_i) q_i^l. = -\int_0^1 \mathrm{d}x q^l(x).$$

Therefore

$$q^{(k)} = \int_0^1 \mathrm{d}q \frac{\mathrm{d}x(q)}{\mathrm{d}q} q^k \quad \Rightarrow \quad P(q) = \frac{\mathrm{d}x(q)}{\mathrm{d}q}$$

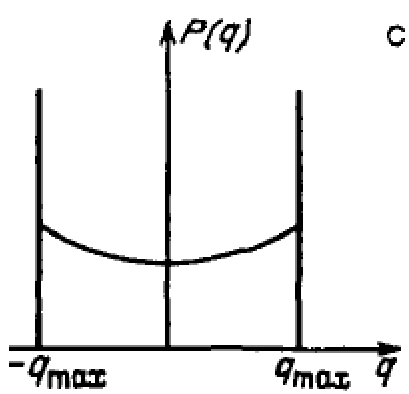
Physical interpretation for q(x)



$x(q) = 2q \quad \text{for} \quad 0 < q < q_1$

$x(q) \in (2q_1 , 1) \quad \text{for} \quad q_1 = q$

$$P(q) = dx/dq = 2\, \theta\,(q_1 - q) \;+\; (1 - 2\,\tau)\; \delta\,(q - q_1)$$



Broad distribution of overlaps $q_{\alpha\beta}$:

Overlap distribution is not a self-averaging quantity

Mezard, M., Parisi, G., Sourlas, N., Toulouse, G. and Virasoro, M. (1984) *Phys Rev. Lett.* **52**, 1156; (1985) *J. Phys. (Paris)* **45**, 843.

$$\overline{P_J(q_1)P_J(q_2)} - \overline{P_J(q_1)}\ \overline{P_J(q_2)} = \frac{1}{3}\left[P(q_1)\delta(q_1 - q_2) - P(q_1)\ P(q_2)\right]$$

The average number of states with a weight in the interval (P, $P + \mathrm{d}P$) (which we denote by $f(P)\ \mathrm{d}P$) can be calculated explicitly [11].

there are an infinite number of states with infinitesimally small weights, but there are also a small number of states with finite weights, for example, the plateau in $q(x)$ (or the δ-function in $P(q)$:

comes from a single state with a minimum value of free energy.

Derrida, B. and Toulouse, G. (1985) *J. Phys. (Paris) Lett.* **46**, L223.

Random free energy hypothesis:

distribution $\rho(f) \propto \exp\left[x_1 N(f - f_c)/T\right]$, $\quad (x_1 < 1)$

Bray, A.J. and Moore, M.A. (1980) *J. Phys.* **C13**, L469.

Boltzmann factor $\exp(-Nf/T)$

Tanaka, F. and Edwards, S.F. (1980) *J. Phys.* **F10**, 2471

The overlaps between three metastable states are not independent.

Joint probability distribution for three states:

$$P(q_{12}, q_{23}, q_{31}) = \lim_{n\to 0} \frac{1}{n(n-1)(n-2)} \sum_{\alpha,\beta,\gamma} \delta(Q_{\alpha\beta} - q_{12})\,\delta(Q_{\beta\gamma} - q_{23})\,\delta(Q_{\alpha\gamma} - q_{31}).$$

An interesting property follows directly from the Parisi ansatz: the probability $P(q_{12}, q_{23}, q_{31}) \neq 0$ only if $q_{ab} = q_{bc} \geqslant q_{ac}$, where (a, b, c) is some permutation of (1, 2, 3). The value of the overlap determines the "distance" $d_{ab} = q(1) - q_{ab}$ between metastable states. Such a space of metastable states with this dista
"ultrametric" space. It can be ima
with the distance between two sta
first common ancestor.

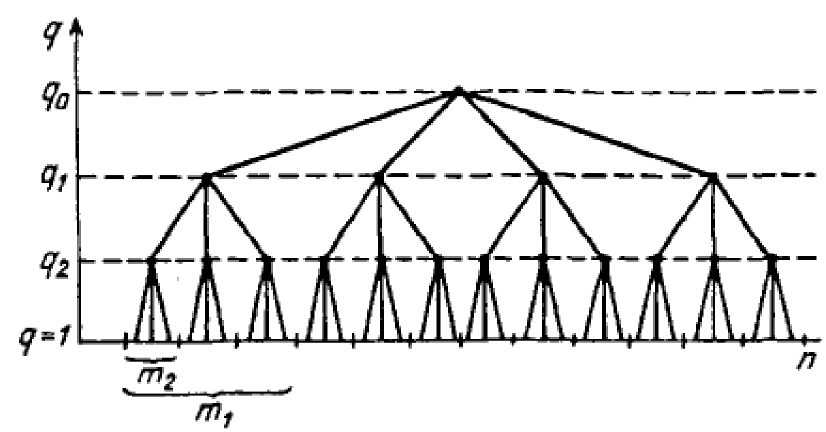


FIG. 14. The definition scheme for the matrix elements Q_{ab} at the two-step replica symmetry breaking.

Hierarchy of pure states

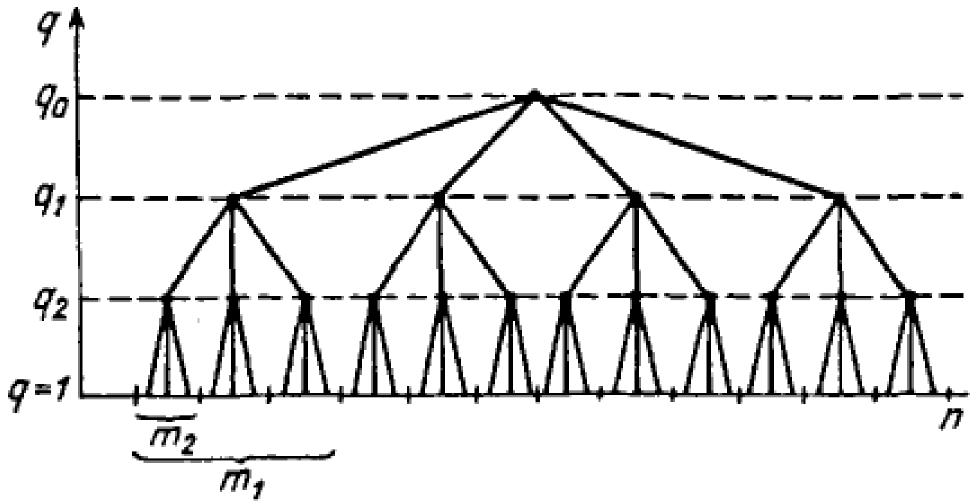


FIG. 14. The definition scheme for the matrix elements Q_{ab} at the two-step replica symmetry breaking.

Ultrametric space:

$d_{ab} = d_{ac} \Rightarrow d_{bc}$

Few problems to solve

1. Derive full equations for q(x) with all terms of order Q^4 and demonstrate that terms with 3 and 4 replica summations are indeed small as extra power of τ

2. Demonstrate that marginal stability condition is fulfilled with accuracy up to higher orders in τ

3. Derive the relation $\chi_{FC} = (1 - q_{EA})/T$ by calculating free energy as function of magnetic field h

4. Find the solution for q(x) function in strong field h >> J

5. Find Paris-type solution for XY glass and for gauge glass

6. Study and present Replica Fourier Transform method developed in arxiv:9709200, arxiv:9703132, Physics Procedia **75**, 2015, P 802