Mean-field SK solution for Spin Glass

D. Sherrington and S. Kirkpatrick, Phys. Rev.Lett. **35**, 1792 (1975)
Phys Rev. B**17**, 4384 (1978)

1. Infinite-range spin glass model: averaging via replica method

2. Edwards-Anderson order parameter

3. Sherrington-Kirkpatrick solution and its inconsistency

4. De Almeida-Thouless instability line

SK model for Spin Glass

$$H = \frac{1}{2} \sum_{i,j} J_{ij} \sigma_i \sigma_j + h \sum_i \sigma_i ,$$

Random couplings J_{ij} are distributed according to

$$P[J_{ij}] = \exp\left[- \frac{(J_{ij} - J_0)^2 N}{2J^2} \right] \left(\frac{N}{2\pi J^2} \right)^{1/2} .$$

Free energy and other physical quantities is to be averaged as

$$F = \int \prod_{i,j} \mathrm{d}J_{ij} \, P[J_{ij}] \, F\{J_{ij}\} ,$$

where

$$F\{J_{ij}\} = -T \ln Z\{J_{ij}\} , \quad Z\{J_{ij}\} = \sum_{\{\sigma_i\}} \mathrm{e}^{-H/T} .$$

Edwards-Anderson order parameter

$$q_{\mathrm{EA}} = \overline{\langle \sigma_i \rangle^2} = (1/N)\ \Sigma_i < \sigma_i >^2$$

It is not a simple object: it contains, first, calculation of thermodynamics average, then, second, taking square, and finally average over disorder realizations

SK solution neglects these subtle issues

Replica Solution of the SK Model

$J_0 = 0$ below

The free energy averaged over random interactions J_{ij} can be represented in the form

$$F = \langle F\{J_{ij}\}\rangle = -T \lim_{n\to 0} \left\langle \frac{1}{n}[Z^n - 1] \right\rangle = -T \lim_{n\to 0} \frac{\langle Z^n\rangle - 1}{n}$$

The nth power of the partition function in (1.2.1) can be obtained as a partition function of the replica Hamiltonian

$$Z^n = \sum_{\{\sigma_i^{(\alpha)}\}} \exp\left(-\frac{H\{\sigma_i^\alpha\}}{T}\right),$$

where

$$H = \frac{1}{2}\sum_{i,j,\alpha} J_{ij}\sigma_i^\alpha\sigma_j^\alpha + h\sum_{i,\alpha}\sigma_i^\alpha,$$

$\alpha = 1, 2, \ldots, n$ labels the replicas.

$$\langle Z^n\rangle = \sum_{\{\sigma_i^\alpha\}} \exp\left[\frac{J^2}{4NT^2}\sum_{i,j,\alpha,\beta}\sigma_i^\alpha\sigma_j^\alpha\sigma_i^\beta\sigma_j^\beta + \frac{h}{T}\sum_{i,\alpha}\sigma_i^\alpha\right] \times \exp\left(\frac{nJ^2N}{4T^2}\right)$$

Here $\alpha \neq \beta$

4-spin term can be decoupled via auxiliary matrix field

$$\langle Z^n \rangle = \exp\left(\frac{nJ^2N}{4NT^2}\right)\int \prod_{\alpha\neq\beta} \mathrm{d}Q_{\alpha\beta} \sum_{\{\sigma\}} \exp\left[-\left(\frac{NQ_{\alpha\beta}^2}{4T^2}\right.\right.$$

$$\left.\left.+\frac{1}{2T^2}Q_{\alpha\beta}\sum_i \sigma_i^\alpha\sigma_i^\beta\right)J^2 + \frac{h}{T}\sum_{\alpha,i}\sigma_i^\alpha\right]\left(\frac{N}{4\pi T^2}\right)^{1/2}.$$

Due to N $\to \infty$ condition steepest descend method can be used:

$$\langle Z^n \rangle = \exp\left(\frac{nJ^2N}{4T^2}\right)\exp\left(-\frac{NJ^2}{4T^2}\sum_{\alpha\neq\beta}Q_{\alpha\beta}^2 - \frac{1}{T}f_s\{Q_{\alpha\beta}\}\right)$$

where

$$f_s\{Q_{\alpha\beta}\} = -T\ln\sum_{\{\sigma_i^\alpha\}}\exp\left[\frac{J^2}{2T^2}\sum_{\alpha\neq\beta}Q_{\alpha\beta}\sigma_i^\alpha\sigma_i^\beta\right],$$

$Q_{\alpha\beta}$ is the saddle-point solution

$$Q_{\alpha\beta} = \langle\sigma^\alpha\sigma^\beta\rangle \equiv \sum_{\{\sigma^\alpha\}}\sigma^\alpha\sigma^\beta\exp\left(\frac{J^2}{2T^2}\sum_{\alpha\neq\beta}Q_{\alpha\beta}\sigma^\alpha\sigma^\beta + \frac{h}{T}\sum_\alpha\sigma^\alpha\right)$$

Sherrington-Kirkpatrick ansatz

simple form of $Q_{\alpha\beta}$ matrix $Q_{\alpha\beta} = q \; (\alpha \neq \beta)$

Now:

$$Q_{\alpha\beta} = \langle \sigma^\alpha \sigma^\beta \rangle \equiv \sum_{\{\sigma^\alpha\}} \sigma^\alpha \sigma^\beta \exp\left(\frac{J^2}{2T^2} \sum_{\alpha \neq \beta} Q_{\alpha\beta} \sigma^\alpha \sigma^\beta + \frac{h}{T} \sum_\alpha \sigma^\alpha \right)$$

Is equivalent to

$$q = \int \frac{dz}{(2\pi)^{1/2}} \sum_{\{\sigma\}} \sigma^\alpha \sigma^\beta \exp\left[\sum_\alpha \sigma^\alpha \left(\frac{h + Jzq^{1/2}}{T} \right) \right] \exp\left(-\frac{z^2}{2} \right) \times \exp\left(-\frac{n}{2T^2} \right)$$

The summation over spin variables of different replicas can be carried out independently, and yields

$$q = \int \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2} \tanh^2 \frac{Jzq^{1/2} + h}{T}.$$

Here z represents random frozen field with unit dispersion

value of q has a break at $T = T_c$: $q = 0$ at $T \geqslant T_c$ and $q = \tau + \frac{1}{3}\tau^2$, $\tau = \frac{T_c - T}{T_c} \ll 1$

Technical point

$$\sum_{\alpha \neq \beta} Q_{\alpha\beta}\sigma_i^{\alpha}\sigma_i^{\beta} = [(\sum_{\alpha} \sigma^{\alpha})^2 - n]\, q$$

The term nq is important to get correct free energy

Free energy for the SK solution:

$$F = -\frac{(1-q)^2}{4T} J^2 - T \int \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2} \ln\left(2\cosh\frac{Jzq^{1/2}+h}{T}\right)$$

Close to T_c and at weak field h free energy per spin is given by

$$f = -\frac{J^2}{4T} + \theta(\tau)\frac{\tau^3}{6} - \frac{1}{2}h^2\frac{1}{T}[1-\theta(\tau)\tau]$$

$$(h, \tau \ll 1).$$

"Third-order" transition

Free energy below T_c is larger than its analytic continuation from the region above T_c

Magnetic susceptibility $\chi = -\partial^2 f/\partial h^2$

has a cusp below T_c

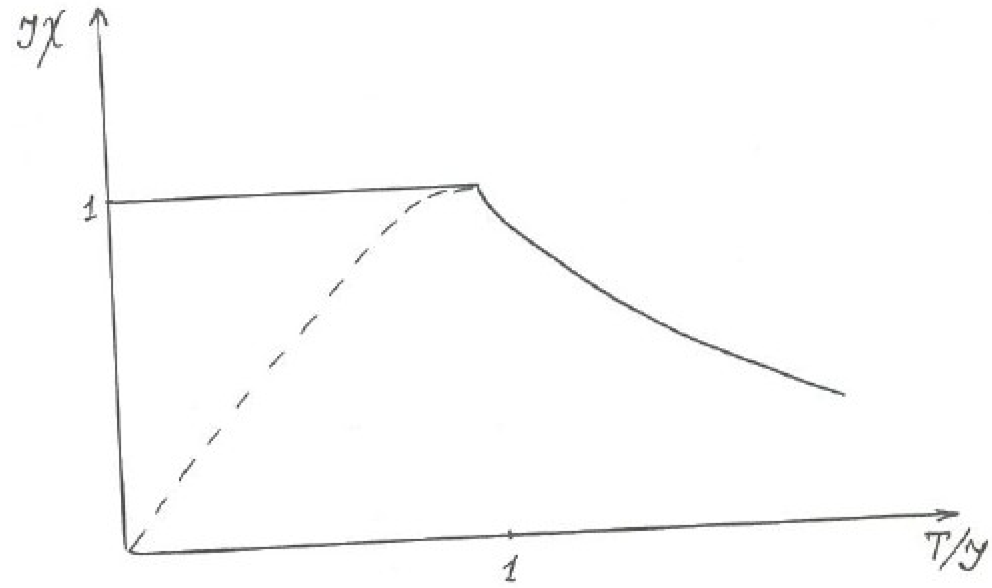


Figure 2 Field-cooling (full line) and zero-field-cooling (dashed line) susceptibilities of the SK model.

SK solution provides a broken line only (ZFC)

The separate result for the FC susceptibility (full line) can be obtained within RSB scheme

Marginal stability condition (TAP): $(T/J)^2 = \langle (1 - m_i^2)^2 \rangle$

Is not fulfilled by SK solution: LHS = $1 - 2\tau + \tau^2$ RHS = $1 - 2\tau + 7/3\ \tau^2$

Zero-temperature entropy catastrophe

Solution of self-consistency equation at T $\rightarrow$ 0 (and h=0)

$$q = \int \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2} \tanh^2 \frac{Jzq^{1/2} + h}{T}$$

Gives $$q = 1 - (2/\pi)^{1/2} T/J + O(T^2)$$

Together with

$$F = -\frac{(1-q)^2}{4T} J^2 - T \int \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2} \ln \left(2 \cosh \frac{Jzq^{1/2} + h}{T} \right)$$

it gives free energy per spin f = T/2π at T $\rightarrow$ 0

Entropy S(0) = - 1/2π

Something is definitely wrong.....

What is wrong with SK solution ?

Physical problem: order parameter q is not unique if many metastable states contribute to Z

Let P_a P_b etc to be probabilities to find our system in **metastable** states a,b... Finite-time dynamics does not perform transitions between these states

$q_{EA} = \Sigma_i (m_i^a)^2 P_a$ EA order parameter within individual minima

In general does not coincide with

$q = \Sigma_i (m_i^a) (m_i^b) P_a P_b$ Full thermodynamic average

Formal proof: instability of the SK solution

Consider $Q_{\alpha\beta} = q + q_{\alpha\beta}$ with small asymmetric $q_{\alpha\beta}$
and study stability w.r.t. these fluctuations

$$F(Q_{\alpha\beta}) = f(q) - \frac{J^2}{8T^4} \sum_{\alpha,\beta,\gamma,\delta} q_{\alpha\beta} q_{\alpha\delta} (\langle \sigma^\alpha \sigma^\beta \sigma^\gamma \sigma^\delta \rangle - \langle \sigma^\alpha \sigma^\beta \rangle \langle \sigma^\gamma \sigma^\delta \rangle),$$

$\langle . . . \rangle$ mean the average over spin variables
with Hamiltonian

$$H = \sum_{\alpha \neq \beta} \sigma^\alpha \sigma^\beta \frac{J^2}{2T^2} q + \Sigma h \sigma^\alpha .$$

$$f(Q_{\alpha\beta}) = f(q) + \frac{J^2}{8nT^4} \left[(2T^2 - c_0) \sum_{\alpha \neq \beta} (q^{\alpha\beta})^2 - 4c_1 \sum_{\alpha,\beta,\gamma} q^{\alpha\beta} q^{\beta\gamma} - c_2 \left(\sum_{\alpha \neq \beta} q^{\alpha\beta} \right)^2 \right]$$

Crucial term

$$c_0 = 2J^2 \int \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2} \operatorname{sech}^4\left(\frac{Jzq^{1/2}+h}{T}\right)$$

$$c_1 = \int \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2} \tanh^2\left(\frac{Jzq^{1/2}+h}{T}\right) \operatorname{sech}^2\left(\frac{Jzq^{1/2}+h}{T}\right),$$

$$c_2 = \int \frac{dz}{(2\pi)^{1/2}} e^{-z^2/2} \tanh^4\left(\frac{Jzq^{1/2}+h}{T}\right) - q^2.$$

c_1 and c_2 do not change their signs with decreasing temperature,

the coefficient $2T^2 - c_0$ does change its sign at $T_c = J$ (for h=0)

Conclusion: replica-symmetric SK solution is unstable at all $T < T_c$

De Almeida – Thouless line

J R L de Almeida†‡ and D J Thouless   J. Phys. A: Math. Gen., Vol. 11, No. 5, 1978. p983

Instability line we found right now coincides with the marginal stability condition

$(T/J)^2 = \langle (1-m_i^2)^2 \rangle$

If external field h>0 is present:

$$(T/J)^2 = \int \frac{\mathrm{d}z}{(2\pi)^{1/2}} \mathrm{e}^{-z^2/2} \operatorname{sech}^4\left(\frac{Jzq^{1/2}+h}{T}\right)$$

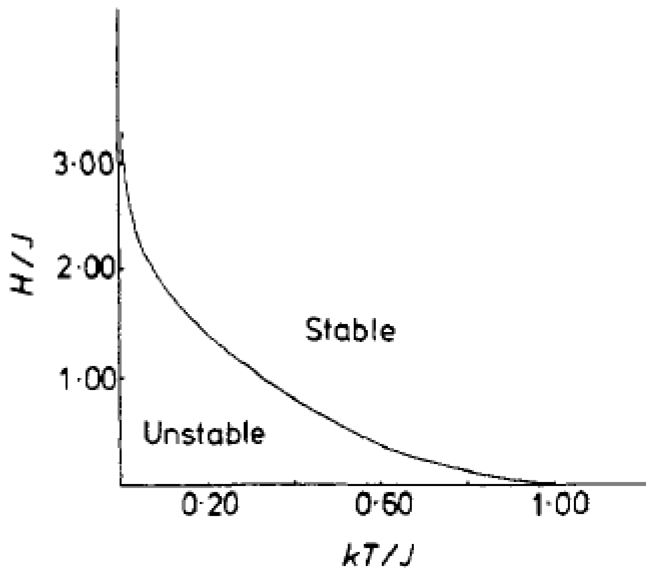
$$\tau_{AT} = \left(\frac{3}{4}\right)^{2/3}\left(\frac{h}{J}\right)^{2/3} \qquad (h \ll J)$$

$$T_{AT} = \frac{4}{3}\frac{J}{(2\pi)^{1/2}}\exp\left(-\frac{h^2}{2J^2}\right) \qquad (h \gg J).$$

$T_{AT} > 0$ for all large $h \gg J$ - how can we understand it ?

The effective field acting on each spin σ_i

consists of the large external field $h \gg J$, T and the field $\tilde{h}_i = \Sigma_j J_{ij}\sigma_j$



there are a small number (ρN)

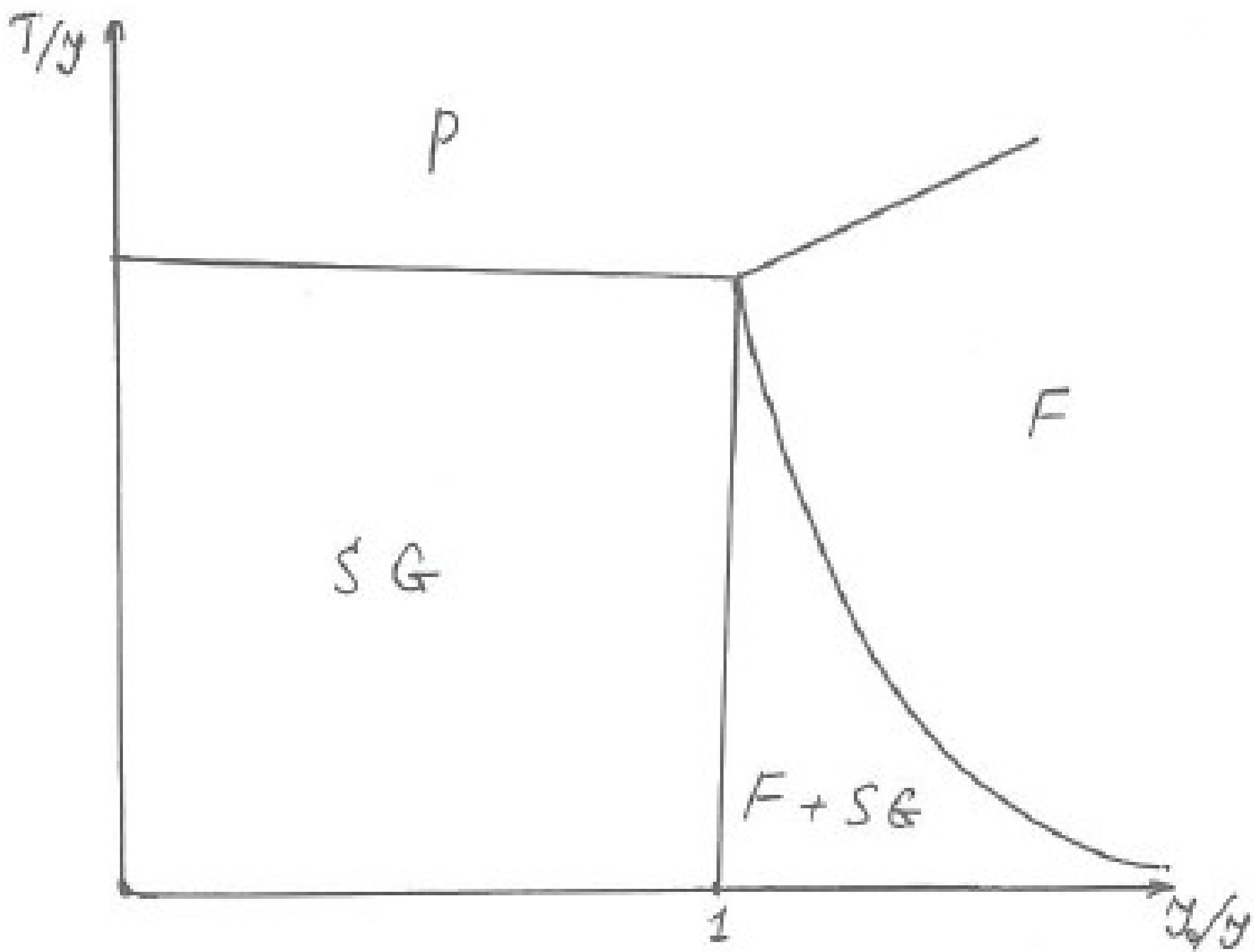
of unfrozen spins

$\rho \approx T \exp\,(-h^2/2J^2)$ /J

$(T_{AT}/J)^2 \sim \rho$

$T_{AT} \sim J \exp\,(-h^2/2J^2)$

General phase diagram



Few problems to solve

- Calculate the shape of the line separating F and F+SG states on the phase diagram of the SK model with nonzero average $J_{0\,.}$ Find it analytically in the limit of large J_0/J and near multicritical point
- Perform the same derivation like SK one for XY spin glass model with random real sign-alternating couplings J_{ij} and for the "gauge glass" model with complex totally random J_{ij} Find transition temperatures and RSB instability increments at near-critical T for both models
- Consider generalization of the SK model with randomly choosen fraction of bonds p = Z/N which are active, while all the rest of bonds is set to zero. Assuming Z to be large, while $N \rightarrow \infty$, try to derive mean-field-like equations and find glass transition point T_g