Spin glass dynamics at $T>T_g$

1. Edwards-Anderson Spin Glass model
   a) Energy functional and dynamic equations with noise
   b) Martin-Siggia-Rose generating functional
   c) Mean-field approximation and equations for the response and correlation functions
   d) Solution near T_g : singularity in $dG/d\omega$

2. Dynamics of 3D spin glasses:
   real experiments and Monte-Carlo studies

3. Model of diffusion on a percolation network and on diluted hypercube

Relaxational dynamics of the Edwards-Anderson model and the mean-field theory of spin-glasses

H. Sompolinsky & Annette Zippelius Phys. Rev. B 25, 6860 (1982)

Order parameter: $$q_{\rm EA} = \lim_{t\to\infty} [\langle S_i(0) S_i(t)\rangle]_J \ .$$

The EA Hamiltonian is $$H = -\sum_{\langle ij\rangle} J_{ij} S_i S_j \ ,$$ spin variables S_i take the values ± 1

Distribution of random J_{ij} $$P(J_{ij}) = (2\pi z/\tilde{J}^2)^{-1/2} \times \exp[-z(J_{ij} - J_0/z)^2/2\tilde{J}^2]$$

Z is the number of nearest neighbours

We consider here a soft-spin version of the EA model defined by

$$\beta H = \frac{1}{2}\sum_{\langle ij\rangle}(r_0\delta_{ij} - 2\beta J_{ij})\sigma_i\sigma_j + u\sum_i \sigma_i^4 + \sum_i h_i\sigma_i, \quad \beta = 1/T \ .$$

To study the relaxational dynamics of spin glasses, we propose a simple phenomenological Langevin equation,

$$\Gamma_0^{-1}\partial_t\sigma_i(t) = -\frac{\delta(\beta H)}{\delta\sigma_i(t)} + \xi_i(t) = \sum_j (r_0\delta_{ij} - \beta J_{ij})\sigma_j(t) + 4u\sigma_i^3(t) + h_i(t) + \xi_i(t) \, .$$

$$\langle \xi_i(t)\xi_j(t')\rangle = \frac{2}{\Gamma_0}\delta_{ij}\delta(t-t') \, ,$$

Objects of interest: pair spin correlation function

$$C_{ij}(t-t') = \langle \sigma_i(t)\sigma_j(t')\rangle$$

and the linear-response function

$$G_{ij}(t-t') = \frac{\partial\langle\sigma_i(t)\rangle}{\partial h_j(t')}, \quad t > t'$$

The FDT reads in the present context

$$C_{ij}(\omega) = \frac{2}{\omega}\mathrm{Im}G_{ij}(\omega)$$

and

$$C_{ij}(t=0) = G_{ij}(\omega=0)$$

$$\mathrm{Re}G_{ij}(\omega) = -\int \frac{d\omega'}{\pi}\frac{\mathrm{Im}G_{ij}(\omega')}{\omega-\omega'}$$

Dynamic generating functional:

P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1978).
C. De Dominicis, J. Phys. (Paris) C 1, 247 (1976); C. De Dominicis and L. Peliti, Phys. Rev. B 18, 353 (1978).

$$Z\{J_{ij},l_i,\hat{l}_i\} = \int D\sigma\, D\hat{\sigma} \exp\left[\int dt\, l_i(t)\sigma_i(t) + i\hat{l}_i(t)\hat{\sigma}_i(t) + L\{\sigma,\hat{\sigma}\} \right]$$

$$L\{\sigma,\hat{\sigma}\} = \int dt \sum_i i\hat{\sigma}_i(t) \left[-\Gamma_0^{-1}\partial_t\sigma_i(t) - r_0\sigma_i(t) + \beta\sum_j J_{ij}\sigma_j(t) - 4u\sigma_i^3(t) - h_i(t) + \Gamma_0^{-1} i\hat{\sigma}_i(t) \right] + V\{\sigma\}$$

The term V, which arises from the functional bian, is given by[32,33] and ensures the proper normalization of Z,

$$Z\{J_{ij},l_i=\hat{l}_i=0\}=1$$

$$V = -\frac{1}{2}\int dt \sum_i \frac{\delta^2(\beta H)}{\delta\sigma_i^2} = -\int dt \sum_i \left[\tfrac{1}{2}r_0 + 6u\sigma_i^2(t)\right]$$

$$\left.\frac{\delta^n\delta^m \ln Z}{\delta\hat{l}_1(\hat{t}_1)\cdots\delta l_m(t_m)}\right|_{l_i=\hat{l}_i=0} = \langle i\hat{\sigma}_1(\hat{t}_1)\cdots\sigma_m(t_m)\rangle_c$$

Response function: $\langle i\hat{\sigma}_j(t')\sigma_i(t)\rangle = G_{ij}(t-t')$ (t > t')

Averaging over J_{ij} is possible since Z = 1

$$[Z]_J \equiv \int \prod dJ_{ij} P(J_{ij}) Z\{J_{ij}\} = \int D\sigma D\hat{\sigma} \exp \Big| L_0\{\sigma,\hat{\sigma}\} + \frac{\beta J_0}{z} \sum_{\langle ij \rangle} \int dt\, i\hat{\sigma}_i(t)\sigma_j(t)$$

$$+ 2\frac{\beta^2 \tilde{J}^2}{z} \sum_{\langle ij \rangle} \int dt\, dt' [i\hat{\sigma}_i(t)\sigma_j(t') i\hat{\sigma}_i(t')\sigma_j(t) + i\hat{\sigma}_i(t)\sigma_j(t) i\hat{\sigma}_j(t')\sigma_i(t')] \Big|$$

here

we use the property $J_{ij} = J_{ji}$.

$$L_0\{\sigma,\hat{\sigma}\} = \int dt \sum_i [i\hat{\sigma}_i(-\Gamma_0^{-1}\partial_t \sigma_i - r_0\sigma_i - 4u\sigma_i^3 - h_i + i\Gamma_0^{-1}\hat{\sigma}_i) + V\{\sigma\} + i\hat{l}_i\hat{\sigma}_i + l_i\sigma_i]$$

Decoupling of the 4-th order terms:

$$[Z]_J = \int \prod_{\alpha}^{4} DQ_\alpha^i(t,t') \exp \Big(-\frac{z}{\beta^2 \tilde{J}^2} \int dt\, dt' \sum_{i,j} (K^{-1})_{ij} [Q_1^i(t,t')Q_2^j(t,t') + Q_3^i(t,t')Q_4^j(t,t')]$$

$$+ \ln \int D\sigma D\hat{\sigma} \exp L\{\sigma,\hat{\sigma},Q_\alpha\} \Big|,$$

where K is the short-range matrix ($K_{ij}=1$ if i,j are nearest neighbors and zero otherwise), and

$$L\{\sigma,\hat{\sigma},Q_\alpha\} = L_0\{\sigma,\hat{\sigma}\} + \tfrac{1}{2} \int dt\, dt' \sum_i [Q_1^i(t,t') i\hat{\sigma}_i(t) i\hat{\sigma}_i(t') + Q_2^i(t,t')\sigma_i(t)\sigma_i(t')$$

$$+ Q_3^i(t,t') i\hat{\sigma}_i(t)\sigma_i(t') + Q_4^i(t,t') i\hat{\sigma}_i(t')\sigma_i(t)].$$

(We have assumed $J_0 = 0$.)

Mean-field limit: z = N (Sherrington-Kirkpatrick model)

One step back:

$$N^{-2}\sum_{i\neq j} i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')\sigma_j(t)\sigma_j(t') = \tfrac{1}{4}N^{-2}\left[\sum_i i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')+\sigma_i(t)\sigma_i(t')\right]^2 - \tfrac{1}{4}N^{-2}\left[\sum_i i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')-\sigma_i(t)\sigma_i(t')\right]^2$$

$$[Z]_J = \int \prod_{\alpha=1}^{4} DQ_\alpha(t,t')\exp\left[-\frac{N}{\beta^2\tilde{J}^2}\int dt\,dt'[Q_1(t,t')Q_2(t,t')+Q_3(t,t')Q_4(t,t')] + \ln\int D\sigma\,D\hat{\sigma}\exp L\{\sigma,\hat{\sigma},Q_\alpha\}\right],$$

$$L\{\sigma,\hat{\sigma},Q_\alpha\} = L_0\{\sigma,\hat{\sigma}\} + \tfrac{1}{2}\int dt\,dt'\sum_i [\,Q_1(t,t')i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')+Q_2(t,t')\sigma_i(t)\sigma_i(t') + Q_3(t,t')i\hat{\sigma}_i(t)\sigma_i(t')+Q_4(t,t')i\hat{\sigma}_i(t')\sigma_i(t)] + O(1)\,.$$

Now $Q_i(t,t')$ (i=1-4) are global variables (no space-dependence)

$$Q_1^0(t,t') = \frac{\beta^2\tilde{J}^2}{2N}\sum_i \langle \sigma_i(t)\sigma_i(t')\rangle\,,$$

$$Q_2^0(t,t') = \frac{\beta^2\tilde{J}^2}{2N}\sum_i \langle i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')\rangle$$

$$Q_3^0(t,t') = \frac{\beta^2\tilde{J}^2}{2N}\sum_i \langle i\hat{\sigma}_i(t')\sigma_i(t)\rangle\,,$$

$$Q_4^0(t,t') = \frac{\beta^2\tilde{J}^2}{2N}\sum_i \langle i\hat{\sigma}_i(t)\sigma_i(t')\rangle$$

$$Q_2^0 = \langle\hat{\sigma}\hat{\sigma}\rangle = 0$$

a vertex $Q_2^0(t,t')\sigma(t)\sigma(t')$ will lead to *violation of causality*, namely, will yield nonzero contributions to $\langle i\hat{\sigma}(t)\sigma(t')\rangle$ with $t>t'$.

$$L\{\sigma_i\hat{\sigma}_i\} = L_0\{\sigma_i,\hat{\sigma}_i\} + \frac{\beta^2\tilde{J}^2}{2}\int dt\,dt'[\,C(t-t')i\hat{\sigma}_i(t)i\hat{\sigma}_i(t') + 2G(t-t')i\hat{\sigma}_i(t)\sigma_i(t')]\,,$$

$$C(t-t') \equiv [\langle \sigma_i(t)\sigma_i(t')\rangle]_J\,,$$

$$G(t-t') \equiv [\langle i\hat{\sigma}_i(t')\sigma_i(t)\rangle]_J$$

$$L_0\{\sigma,\hat{\sigma}\} = \int dt \sum_i [i\hat{\sigma}_i(-\Gamma_0^{-1}\partial_t\sigma_i - r_0\sigma_i - 4u\sigma_i^3 - h_i + i\Gamma_0^{-1}\hat{\sigma}_i) + V\{\sigma\} + i\hat{l}_i\hat{\sigma}_i + l_i\sigma_i]$$

The new effective bare propagator is

$$G_0^{-1}(\omega) = r_0 - i\omega\Gamma_0^{-1} - \beta^2\tilde{J}^2 G(\omega)$$

and the effective noise ϕ is a Gaussian random variable with width

$$\langle \phi_i(\omega)\phi_i(\omega')\rangle = [2\Gamma_0^{-1} + \beta^2\tilde{J}^2 C(\omega)]\,\delta(\omega+\omega')$$

$$\sigma_i(\omega) = G_0(\omega)[\phi_i(\omega) + h_i(\omega)] - 4uG_0(\omega)\int d\omega_1 d\omega_2 \sigma_i(\omega_1)\sigma_i(\omega_2) \times \sigma_i(\omega-\omega_1-\omega_2)\,.$$

DYNAMICS FOR $T \geq T_c$

Key quantity: $$\Gamma^{-1}(\omega) = i\frac{\partial G^{-1}(\omega)}{\partial \omega}$$ Effective kinetic coefficient

The dynamic-response function obeys a Dyson equation

$$G^{-1}(\omega) = G_0^{-1}(\omega) + \Sigma(\omega) ,$$

$$G_0^{-1}(\omega) = r_0 - i\omega\Gamma_0^{-1} - \beta^2\tilde{J}^2 G(\omega)$$

the nonlinear coupling u.

$$\Gamma^{-1}(\omega) = \frac{\Gamma_0^{-1} + i\frac{\partial \Sigma}{\partial \omega}}{1 - \beta^2\tilde{J}^2 G^2(\omega)}$$

random exchange

assume that $$\frac{\partial}{\partial \omega}\mathrm{Im}\,\Sigma(0) \equiv \frac{\partial}{\partial \omega}\mathrm{Im}\,\Sigma\Big|_{\omega=0}$$ is finite (it will be proven later)

Then $$\Gamma^{-1}(\omega=0) = \frac{\Gamma_0^{-1} + \mathrm{Im}\frac{\partial \Sigma}{\partial \omega}(0)}{1 - \beta^2\tilde{J}^2 G^2(0)} .$$ diverges at $T_c = \tilde{J}G(0)$

$$\Gamma^{-1}(\omega=0)=\frac{\Gamma_0^{-1}+\mathrm{Im}\,\frac{\partial\Sigma}{\partial\omega}(0)}{1-\beta^2\tilde{J}^2G^2(0)}\ .$$

As T approaches T_c, Γ^{-1} $(\omega=0)$ shows critical slowing down,

$$\Gamma^{-1}(0)\propto\tau^{-1}\ \text{where}\ \tau=\frac{T}{T_c}-1$$

Now recall FDT: $C_{ij}(t=0)=G_{ij}(\omega=0)$

For i=j we use obvious relation for Ising spins: C(t=0) = 1
and conclude that G(ω=0)=1 leading to

$$T_c=\tilde{J}$$

Now we can solve for the small difference g(ω) = G(ω)-1 << 1

$$g^2(\omega)+2\tau g(\omega)+i\omega\tilde{\Gamma}_0^{-1}=0$$

$$\frac{1}{\tilde{\Gamma}_0}=\frac{1}{\Gamma_0}+\Im\frac{\partial\Sigma(\omega)}{\partial\omega}(0)$$

$$g(\omega)=-\tau+\sqrt{\tau^2-i\omega\tilde{\Gamma}_0^{-1}}$$

$$G(t)=\frac{1}{2}\left(\frac{\Gamma}{\pi}\right)^{1/2}\frac{1}{t^{3/2}}\exp\left(-\frac{t}{t_0}\right)\theta(t)$$

where $t_0=\Gamma\tau^{-2}$

Self-energy $\Sigma(\omega)$

1) We do not need calculation of $\Sigma(0)$ in order to find T_c - rather, we can use FDT and the condition $S^2=1$ to find $\Sigma(0)$:

$$1 + \beta^2 J^2 = r_0 + \Sigma(0)$$

2) We should check the assumption of finite $\frac{\partial}{\partial\omega}\mathrm{Im}\,\Sigma(0) \equiv \frac{\partial}{\partial\omega}\mathrm{Im}\,\Sigma\Big|_{\omega=0}$

$$\frac{\partial\Sigma(0)}{\partial\omega} = 2(12u)^2 \int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi} C(\omega_1)C(\omega_1-\omega_2)\frac{\partial}{\partial\omega_2}\mathrm{Im}\,G(\omega_2)$$

At T_c, $G(\omega)\sim\omega^{1/2}$ and $C(\omega)\sim\omega^{-1/2}$

Thus the above integral is indeed infra-red- finite (with cutoff at high frequency)

Dynamics of 3D Spin Glass

Dynamics of three-dimensional Ising spin glasses in thermal equilibrium **Andrew T. Ogielski**

Phys Rev B 32, 7384 (1985)

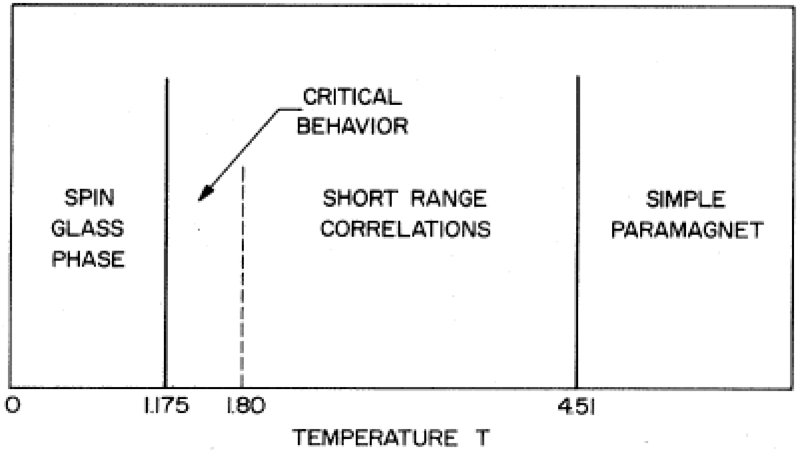


FIG. 1. Graphical representation of distinct temperature regimes observed in the three-dimensional Ising spin glass.

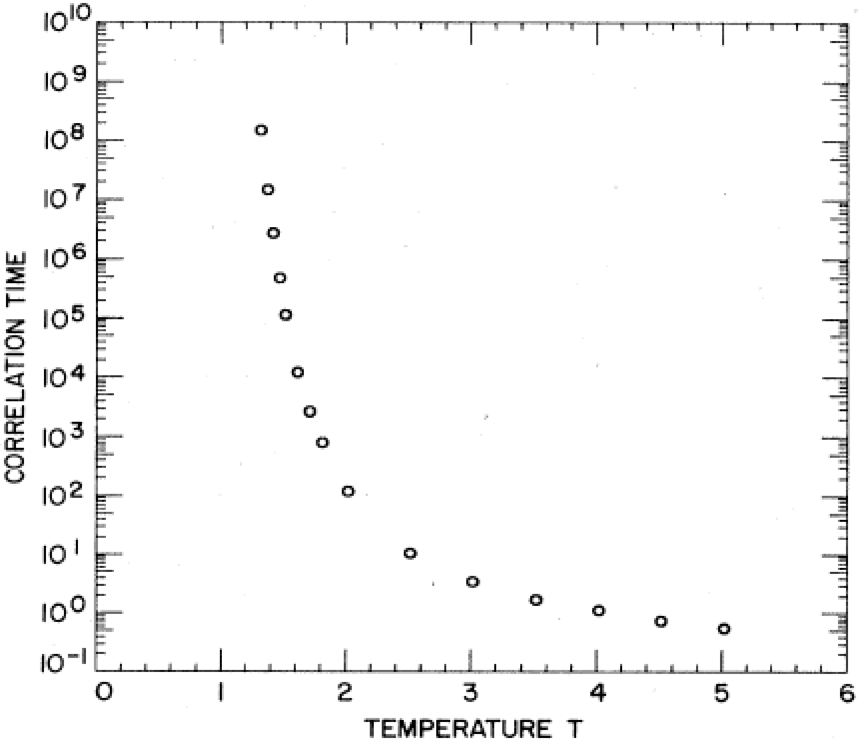
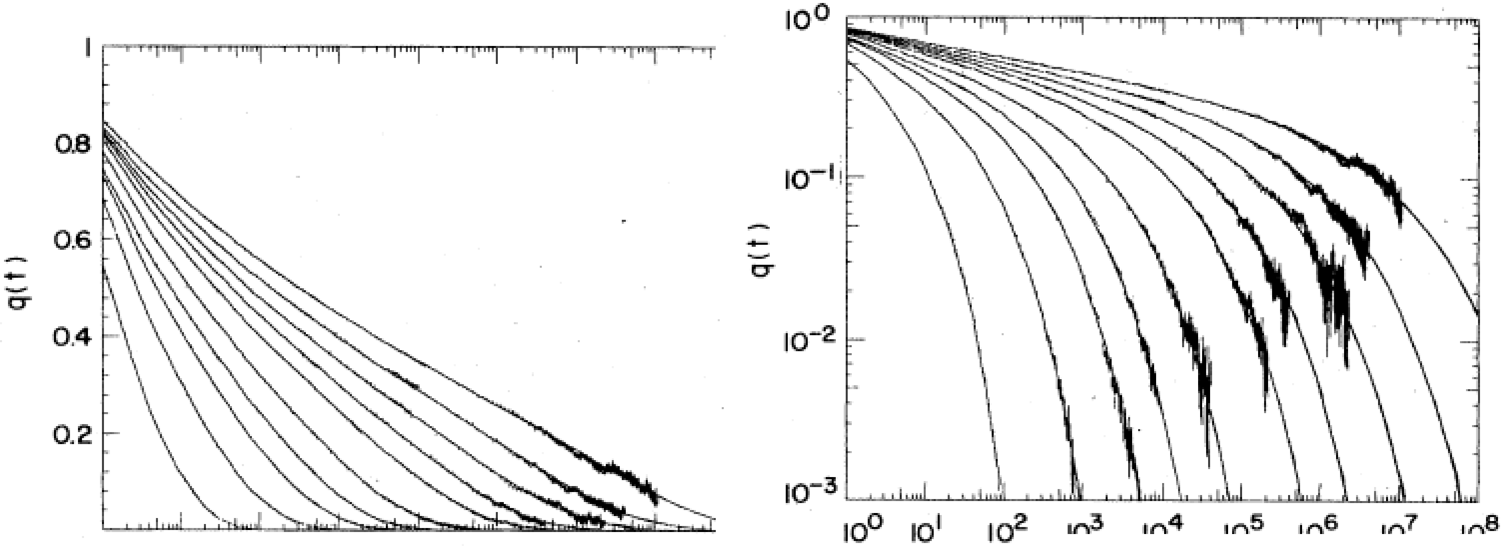


FIG. 3. Temperature dependence of the correlation time τ,



$$q(t)=c\frac{e^{-\omega t^{\beta}}}{t^{x}}$$

FIG. 7. Dynamic correlation functions $q(t)$ above T_g. Short-time behavior is well seen in the semilogarithmic plot (top), long-time behavior can be seen only in the log-log plot (bottom). Data points are shown together with error bars. From left to right, the temperatures are $T=2.50$, 2.00, 1.80, 1.70, 1.60, 1.50, 1.45, 1.40, 1.35, and 1.30. Lattice size 64^3.

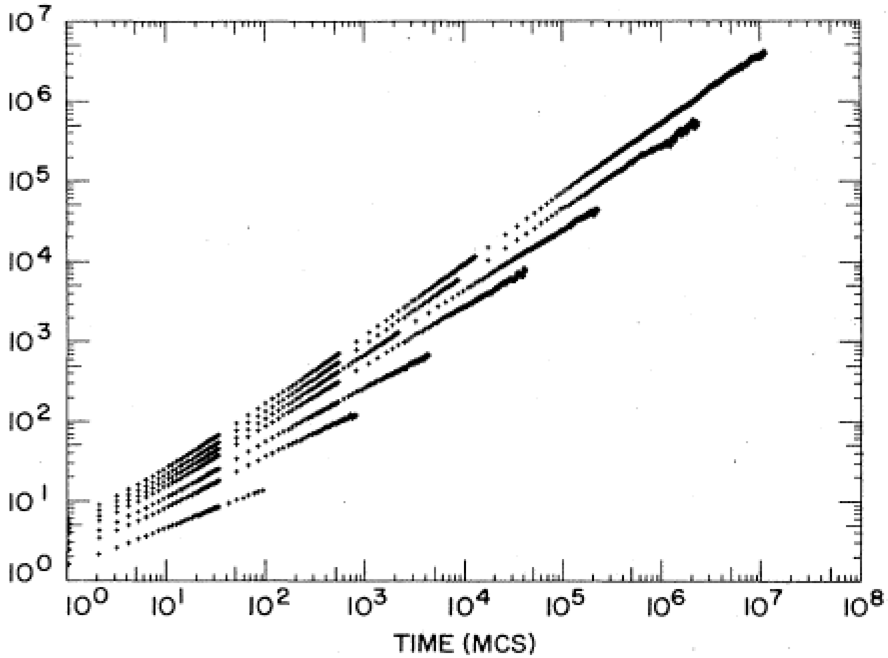


FIG. 10. Correlation functions $q(t)$ shown before in Fig. 7 are converted into a plot of $-t/\ln q(t)$ vs t on the log-log scale. Data points would appear as horizontal lines if $q(t)\sim\exp(-t/\tau)$; this is not seen here. Asymptotically straight lines seen in the graph indicate the Kohlrausch behavior $\exp(-\omega t^{\beta})$ instead, with $\beta<1$. The temperatures are $t=2.50$ (bottom), 2.00, 1.80, 1.60, 1.50, 1.40, and 1.30 (top).

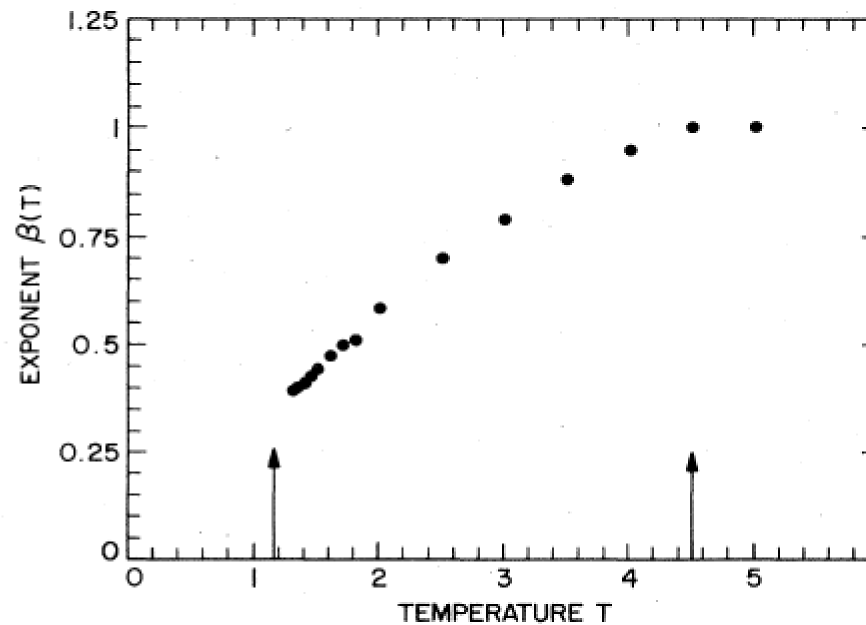


FIG. 11. Temperature dependence of the exponent β defined in Eq. (13). The arrows mark the spin-glass transition temperature T_g and the Curie point T_c of nonrandom Ising model.

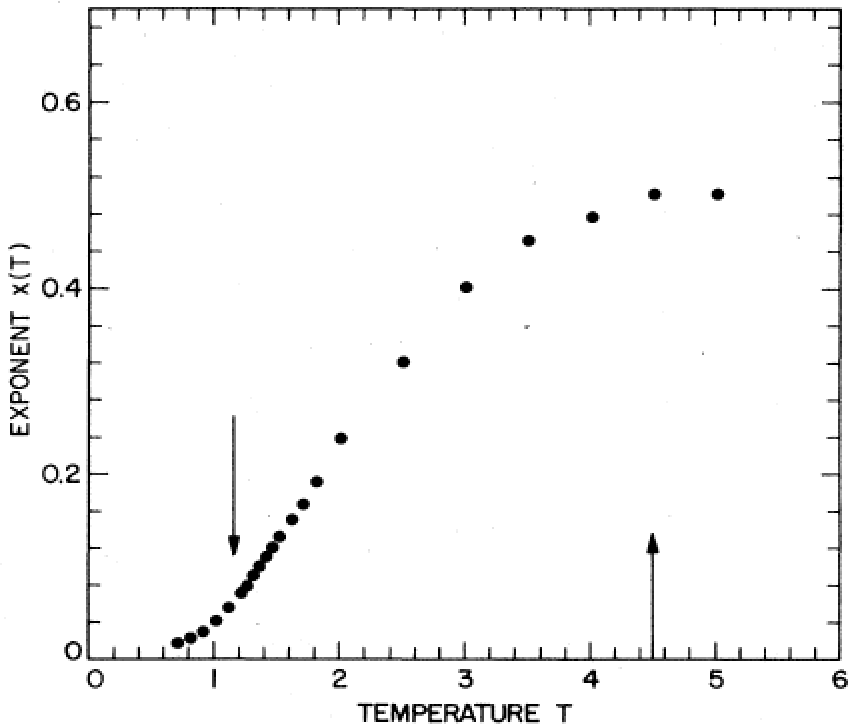


FIG. 12. Temperature dependence of the exponent x defin by Eq. (13) above T_g, and determined from the algebraic decay of $q(t)$ around and below T_g. The arrows mark T_g and T_c as in Fig. 11.

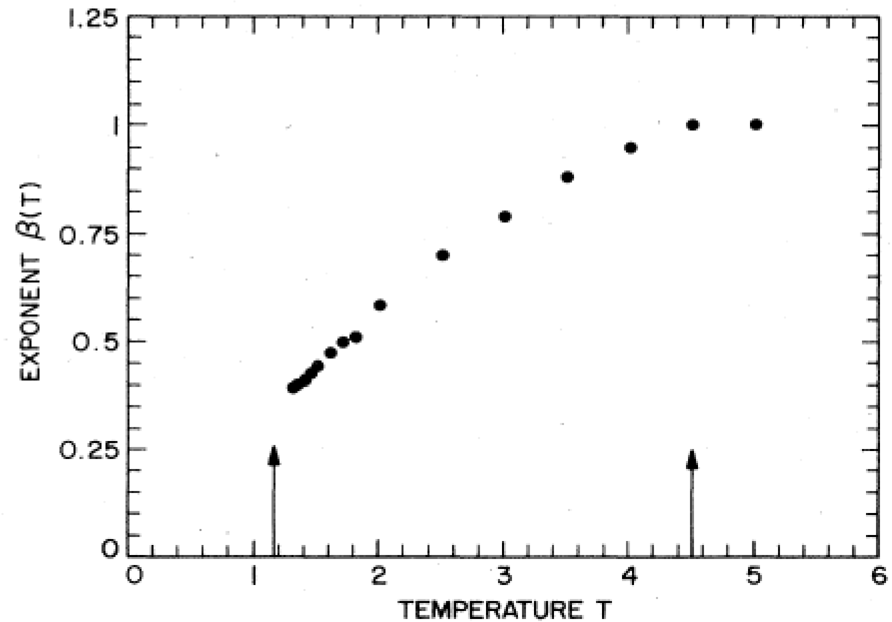


FIG. 11. Temperature dependence of the exponent β defined in Eq. (13). The arrows mark the spin-glass transition temperature T_g and the Curie point T_c of nonrandom Ising model.

$$q(t) \approx t^{-x} Q(t/\tau) ,$$

$$x = \frac{1}{2} \left[\frac{d-2+\eta}{z} \right] ,$$

$$\tau \approx (T - T_g)^{-z\nu} .$$

$$T_g = 1.175 \pm 0.025$$

$$\nu = 1.3 \pm 0.1,$$

$$z = 6.0 \pm 0.8$$

$$\tau = \int_0^\infty dt\, t q(t) \Big/ \int_0^\infty dt\, q(t)$$

Comparison with MF model

MF model

1) exponential relaxation above T_g
2) exponent $z\nu = 2$
3) exponent $x = 1/2$

3D Ising SG (Ogielski MC)

1) stretched exponential, $1/3 < \beta < 1$
2) exponent $z\nu \approx 8$
3) exponent $x < 0.1$ at T_g

Real experiments

COMBINED THREE-DIMENSIONAL POLARIZATION ANALYSIS AND SPIN ECHO STUDY OF SPIN GLASS DYNAMICS

Journal of Magnetism and Magnetic Materials 14 (1979) 211–213

F. MEZEI and A.P. MURANI

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spin correlation function $S(\kappa, t)$ for a Cu–Mn spin glass alloy   a single scan in the range $10^{-12} < t < 10^{-9}$ s.

neutron spin echo and polarization analysis

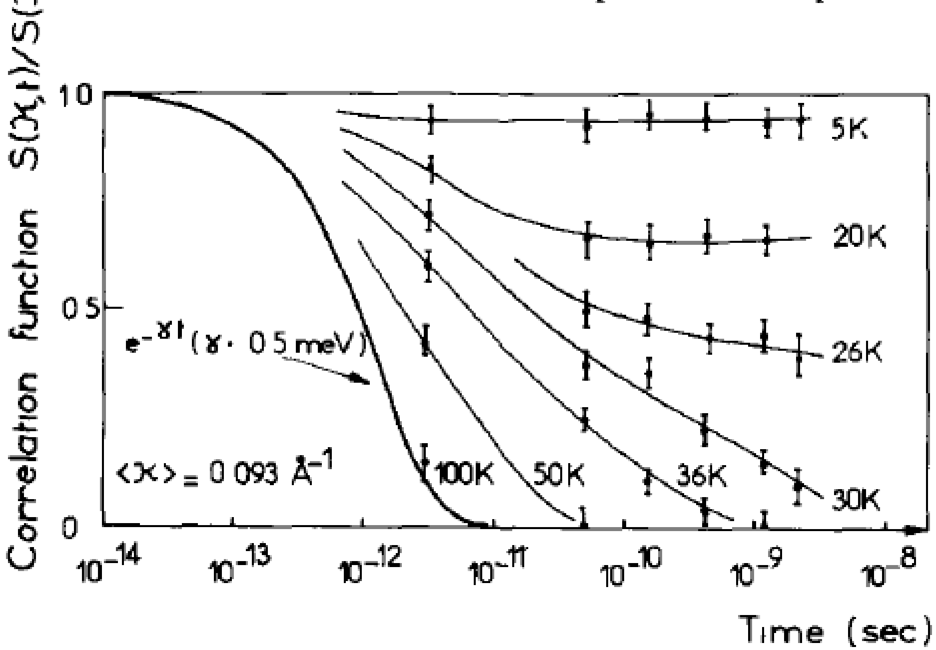


Fig. 3. The measured time dependent spin correlation function for Cu–5 at% Mn at various temperatures The thick line corresponds to the simple exponential decay. The thin lines are guides to the eye only

Again looks like stretched exponential in a broad range of T

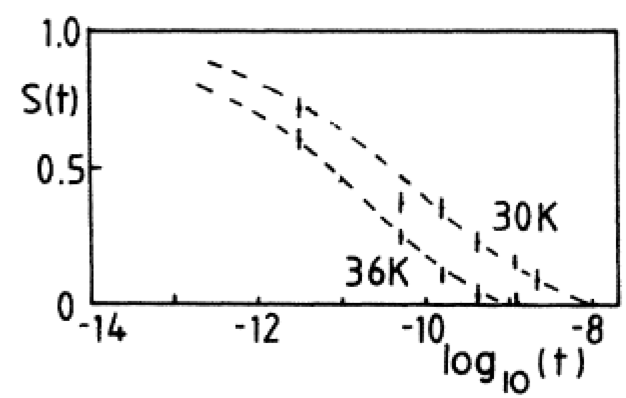


FIG. 3. Neutron spin-echo data from Ref. 11 on Cu–5 at.% Mn at two temperatures just above $T_g = 28.5$ K. The curves are stretched exponential fits with $\beta = 0.33$ and 0.37.

Dynamic scaling in the $Eu_{0.4}Sr_{0.6}S$ spin-glass

N. Bontemps and J. Rajchenbach R. V. Chamberlin and R. Orbach

Phys Rev B 30, 6514 (1984)

frequency range (10^{-2}—10^5 Hz)

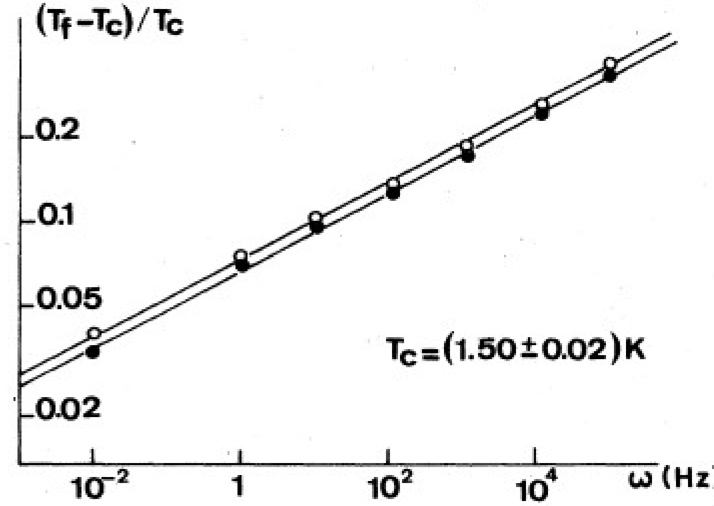
In order to cover the largest range of frequencies, we have used two different techniques for measuring χ'' and χ' on a single sample. The high-frequency regime (10—10^5 Hz) has been investigated at ESPCI by measuring the magnetization (or susceptibility) using a Faraday rotation method.[7,12] The low-frequency regime (10^{-2}—10 Hz) has been investigated at UCLA using a SQUID magnetometer.[13] For purely technical reasons,

$$\tau/\tau_0 \propto \xi_{EA}^z \propto [(T-T_c)/T_c]^{-z\nu} . \quad (5)$$

Equation (5) defines a zero-field freezing temperature associated with the frequency ω taking $\tau \sim 1/\omega$:

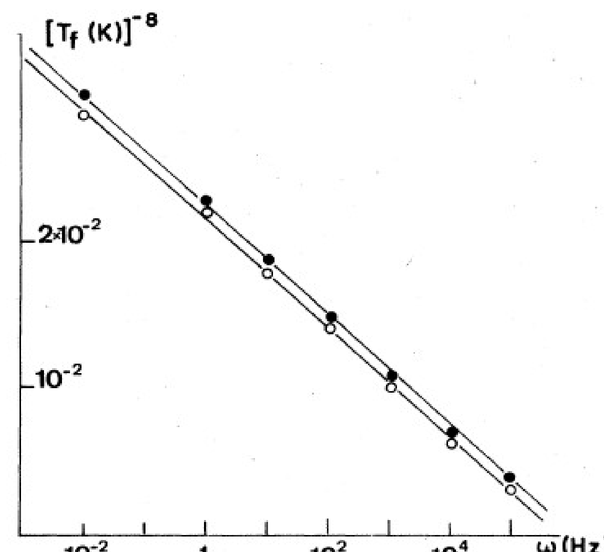
$$\omega/\omega_0 \propto [(T-T_c)/T_c]^{z\nu} . \quad (6)$$

$z\nu = 7.2 \pm 0.5$



$\tau_0 \sim 2 \times 10^{-7}$ s, $T_c \neq 0$

Another fit: $T_c = 0$ $z\nu = 8 \pm 0.5$ $\tau_0 \sim 10^{-5}$ s



Model calculations for the stretched-exponent:

Random walks on a closed loop and spin glass relaxation

I. A. Campbell

J. Physique Lett. **46** (1985) L-1159 - L-1162

Random walks on a hypercube and spin glass relaxation

I A Campbell†, J M Flesselles‡, R Jullien† and R Botet†

J. Phys. C: Solid State Phys. **20** (1987) L47–L51.
Phys Rev B 37, 3825 (1988)

The calculations were done on hypercubes of dimension 10, 14, 16 and 17

$$\langle r^2(t)\rangle = \sum_{i=1}^{d}(x_i(t) - x_i(0))^2 \qquad q(t) = \langle r_\infty^2\rangle - \langle r^2(t)\rangle \propto \exp[-(t/\tau)^\beta]$$

65 536 vertices

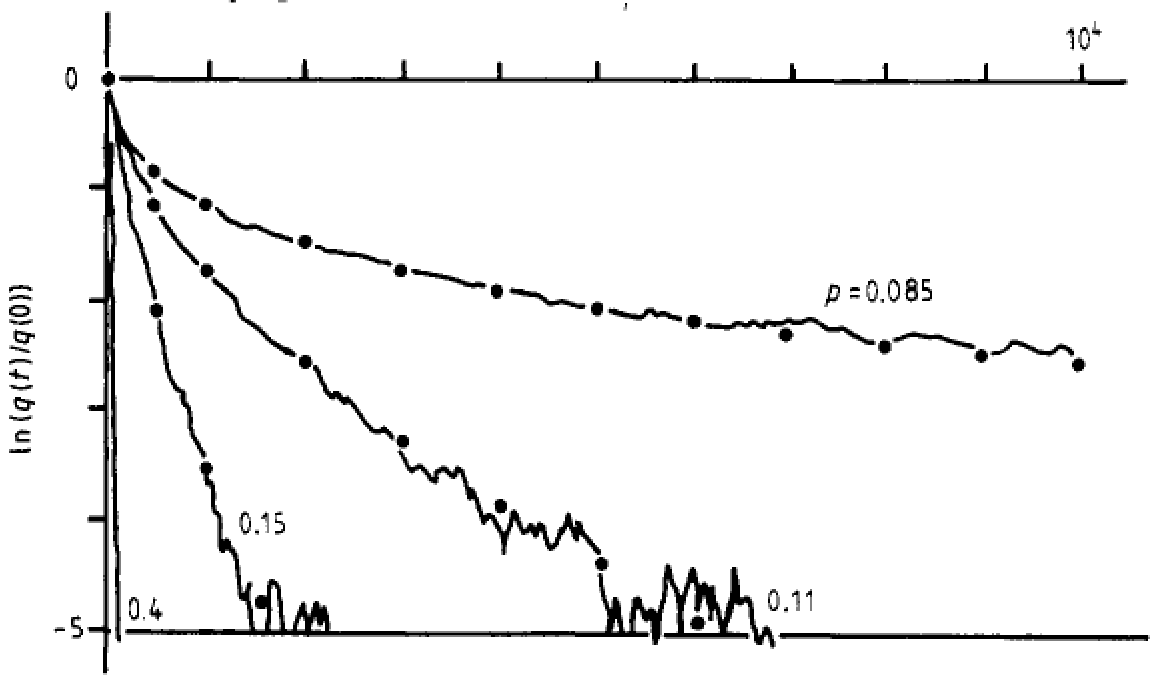


Figure 3. Selected results for $q(t)$ as a function of t at different concentrations p for $d = 16$. The points indicate best-fit curves of the form (1).

Critical value of β

$\beta_c = 1/3$

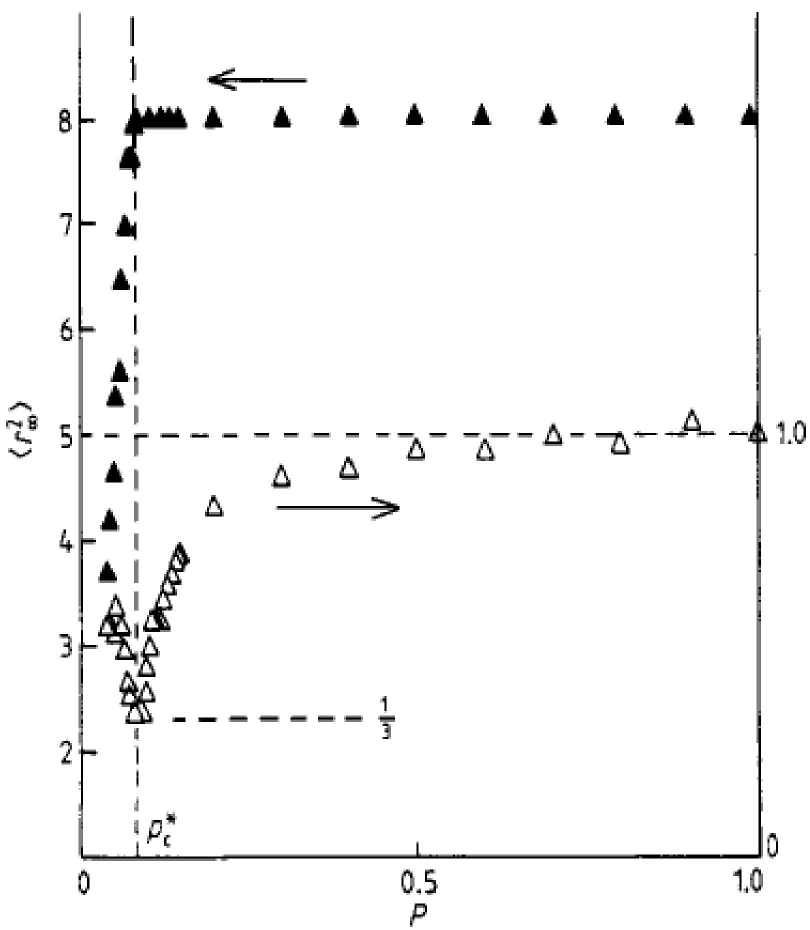


Figure 1. $\langle r_\infty^2 \rangle$ and β as functions of the concentration p for the 16-dimensional hypercube. The threshold concentration p_c^* is indicated by a broken line.

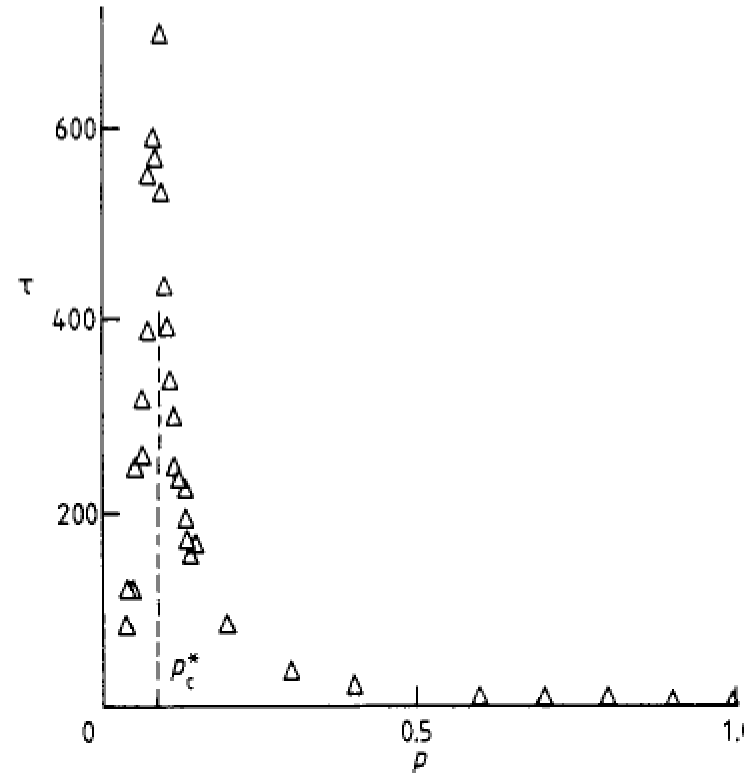


Figure 2. The relaxation rate parameter τ as a function of concentration p for the 16-dimensional hypercube. p_c^* is again indicated by a broken line.

d	p_c^*	$\beta(p_c^*)$	p_c'
10	0.16	0.44	0.148
14	0.095	0.38	0.085
16	0.085	0.34	0.073
17	0.075	0.335	0.069

Reasonable (and unsolved) theoretical model

$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \ ,$$

Sparse matrix model

Summation goes over all pairs (ij) but matrix J_{ij} is strongly diluted:
with a probability (1-p) the bond is erased, $p = Z/N \ll 1$

Static version of the same problem was studied (among others) in:

"Mean-field theory of Spin Glasses with finite Coordination number"
I.Kanter and H.Sompolinsky, Phys.Rev.Lett. **58**, 164 (1987)

Such a model contains strong statistical fluctuations (like real 3D SG)
but neglects thermodynamic fluctuations (long-wave-length modes)
Thus it can be useful for the description of the Griffiths phase in 3D
(but is not useful to study the critical exponents)

Few problems to solve

- Look through the literature to find more recent experimental data on the spin glass dynamics (real or computer experiments). Try to compare findings (if any) with old data. Both Griffiths region and critical region are of interest.
- Derive the value of the exponent $\beta = 1/3$ for the percolation threshold on the hypercube using the paper by S.Alexander & R.Orbach, J.Physique 43, L625 (1982). Compare with the paper by A.Bray and G.Rogers, Phys Rev B 38, 11461 (1988)
- Try to derive (and then solve) the system of integral equations which describe dynamics of the Ising spin glass on a Random Regular Graph with finite connectivity Z