(Ising SG to be discussed here)

Critical hierarchy of modes in 3D spin glasses

Lecture based on L.Ioffe & M.Feigel'man results (1985-1990) + new analysis

Hierarchical structure of an Edwards-Anderson spin glass

Zh. Eksp. Teor. Fiz. **89**, 654–679 1985

V.S. DOTSENKO, M.V. FEIGEL'MAN and L.B. IOFFE

SPIN GLASSES AND RELATED PROBLEMS

Sov. Sci. Rev. A. Phys. Vol. 15, 1990, pp. 1-250

1. Thouless-Anderson-Palmer method for a finite-range Spin Glass: selection of slow modes and eigenfunctions of the J_{ij} matrix

2. Effective Hamiltonian of slow modes: parity-based classification

3. Formation of super-paramagnetic clusters and their interactions

4. Discrete RG transformation and critical hierarchy

TAP method for a finite-range Spin Glass: selection of slow modes and eigenfunctions of the J_{ij} matrix

D.Thouless, P.Anderson & R.Palmer 1977:

$$F = -\frac{1}{2}\sum_{i,j} J_{ij}m_i m_j - \frac{1}{4T}\sum_{i,j} J_{ij}^2(1-m_i^2)(1-m_j^2) + \frac{T}{2}\sum_i \left[(1+m_i)\ln\frac{1+m_i}{2} + (1-m_i)\ln\frac{1-m_i}{2}\right]$$

Near T_c expand F in powers of m_i then use basis of eigenfunctions of J_{ij}

$$m_i = \sum_\lambda a_\lambda \psi_\lambda(i), \quad \sum_j J_{ij}\psi_\lambda(i) = E_\lambda \psi_\lambda(i).$$

For

$$Z = N \to \infty \quad \rho(E) = (2\pi)^{-1}(4-E^2)^{1/2}\theta(4-E^2), \quad m_i = a_0\psi_0(i) + \delta m_i = a_0\psi_0(i) + \sum_{\alpha\neq 0} a_\alpha \psi_\alpha(i),$$

$$F = \frac{1}{2}\tau^2 q + \frac{1}{2}\tau q^2 + \frac{1}{2}q^3 + \frac{1}{3}\sum_i a_0^3\psi_0^3(i)\left[\sum_{\alpha\neq 0} a_\alpha\psi_\alpha(i)\right] + \frac{1}{2}\sum_{\alpha\neq 0}(\tau^2+2-E_\alpha)a_\alpha^2,$$

$\boxed{\tau = T - T_0} = T - 1$ and $q = a_0^2/N$

$$F\{a_0\} = \frac{1}{2}\tau^2 q + \frac{1}{2}\tau q^2 + \frac{1}{2}q^3 - \frac{1}{18}\sum_{i,j} a_0^3\psi_0^3(i)g(i,j)a_0^3\psi_0^3(j), \qquad g_{ij} = \delta_{ij}\int \frac{\rho(E)\,dE}{2-E+\tau^2} = \delta_{ij}$$

$$F\{a_0\} = \frac{1}{2}\tau^2 q + \frac{1}{2}\tau q^2 + \frac{1}{2}q^3 \quad - \frac{1}{18}\sum_{i,j} a_0^3\psi_0^3(i) g(i,j) a_0^3\psi_0^3(j),$$

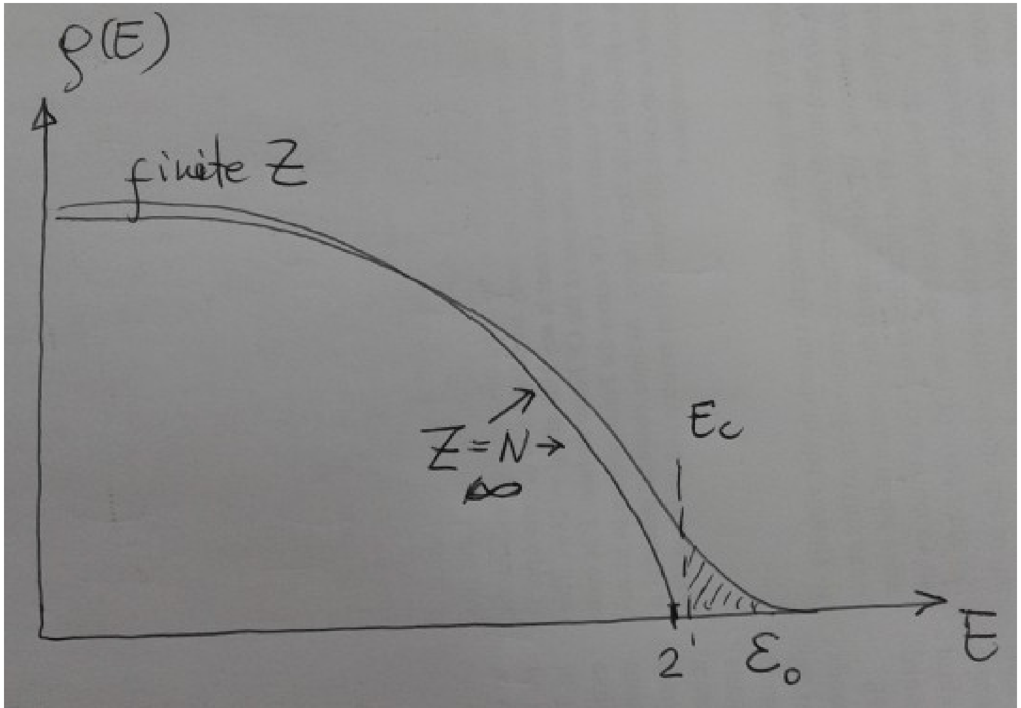
$$g_{ij} = \delta_{ij}\int \frac{\rho(E)\,\mathrm{d}E}{2 - E + \tau^2} = \delta_{ij} + O(\tau).$$

$\Sigma_j\, g_{ij}\, \psi_0(j) = 0$ Orthogonality

TAP expression for the free energy: $F\{a_0\} = \frac{1}{6}(\tau + q)^3 - \frac{1}{6}\tau^3$ $\quad q = \frac{a_0^2}{N} = |\tau|$

Generalization for finite large Z (coordination number)

$\overline{J_{ij}^2} = cK(r_i - r_j)$ $\quad \int d^3 r K(r) = J_0^2$ $\quad Z = cK^{-3/2}[\int K(r) r^2\, d^3 r]^{3/2} \gg 1$



Below $c = J_0 = 1$

$\varepsilon_0 \sim Z^{-4/3}$

E_c is the mobility edge

$$\rho(E) \sim \exp[-a_1(\epsilon/\epsilon_0)^{3/4}].$$

Tail of localized states

3D finite-range hopping: analysis of the spectrum

$$Z = cK^{-3/2}[\int K(r) r^2\, d^3r]^{3/2} \gg 1$$

$\overline{J_{ij}^2} = cK(r_i - r_j)$ $\quad \int d^3r K(r) = J_0^2 \quad$ $c = J_0 = 1$

$$\left.\begin{aligned} \rho(E) &= -\frac{1}{\pi} \operatorname{Im} \lim_{l\to 0} \int \mathscr{D} Q^{\alpha\beta} [(E+i\delta)\delta_{\alpha\beta} + Q^{\alpha\beta})^{-1} e^{S\{Q\}}, \\ S\{Q\} &= -\frac{1}{4} \sum_{i,j,\alpha,\beta} Q_i^{\alpha\beta} K^{-1}(r_i - r_j) Q_j^{\alpha\beta} \\ &\quad -\frac{1}{2} \operatorname{Tr} \ln [(E+i\delta)\delta_{\alpha\beta} + Q_i^{\alpha\beta}]. \end{aligned}\right\} \quad (4.A.4)$$

$$\int d^3r K(r) = J_0^2$$

$K^{-1}(r_i - r_j)$ can be replaced by $\left(1 - \frac{\nabla^2}{\kappa^2}\right) \delta(r_i - r_j)$

saddlepoint approximation $\quad \boldsymbol{Q} = -[\boldsymbol{Q} + \mathbf{1}(E+i\delta)]^{-1}$, the diagonal solution: $Q^{\alpha\beta} = \delta^{\alpha\beta} Q_0$, with

$$Q_0 = \begin{cases} -\frac{1}{2}E + i(1 - \frac{1}{2}E^2)^{1/2} & (|E| \leqslant 2) \\ -\frac{1}{2}E + \operatorname{sgn}(E)\,(\frac{1}{4}E^2 - 1)^{1/2} & (|E| \geqslant 2) \end{cases}$$

$$\boldsymbol{Q} = Q_0 \mathbf{1} + \tilde{\boldsymbol{Q}}.$$

$$S = \operatorname{Tr} \int \left\{ \tilde{\boldsymbol{Q}} \left[i(-\epsilon)^{1/2} - \frac{1}{2\kappa^2} \nabla^2 \right] \tilde{\boldsymbol{Q}} + \frac{1}{6} \tilde{\boldsymbol{Q}}^3 \right\} d^3x.$$

$$\delta G = G(\delta\Sigma)G, \quad \delta\Sigma = \frac{\kappa^3}{2\pi} e^{i\pi/4} (-4\epsilon)^{-1/3}$$

the corrections become important at $\boxed{-\epsilon \leqslant \epsilon_0 = \frac{\kappa^4}{(4\pi)^{4/3}} \approx Z^{-4/3}}$

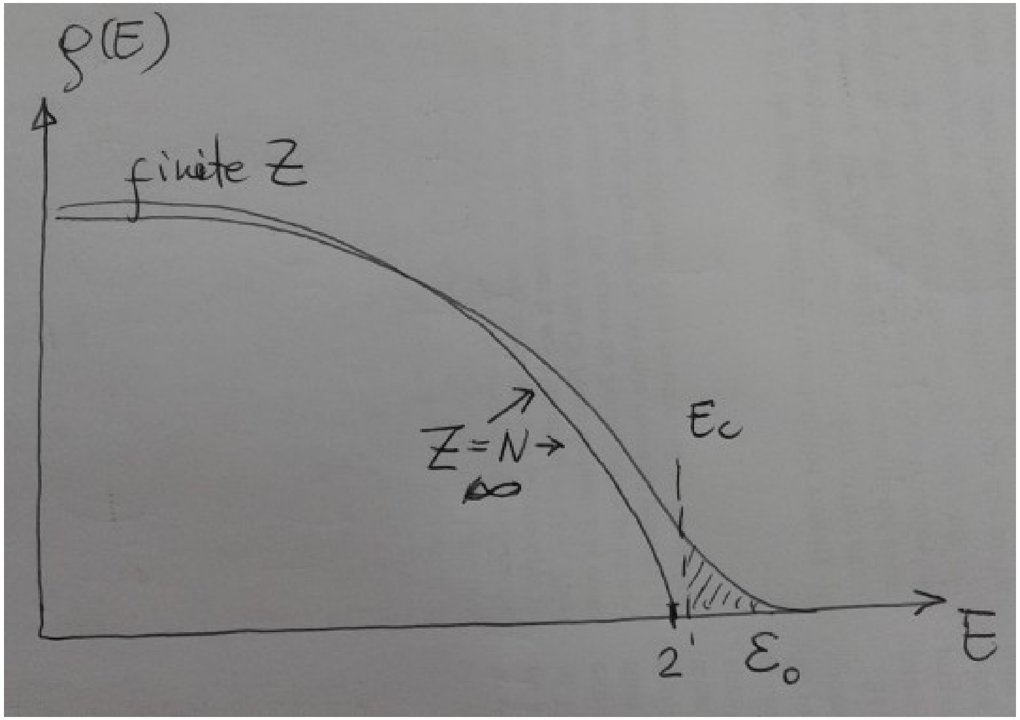
Expansion over eigen-modes for finite large Z

$$m_i = \sum_\lambda a_\lambda \psi_\lambda(i) + \delta m_i = \sum_\lambda a_\lambda \psi_\lambda(i) + \sum_\alpha a_\alpha \psi_\alpha(i)$$

$$\tilde{m}_i = \sum_\lambda a_\lambda \psi_\lambda(i),$$

where we denote "slow" modes by a subscript λ and "fast" ones by α.

$$l(\epsilon) \sim Z^{1/3}\epsilon^{-1/4}. \qquad \rho(\epsilon_0) \approx \epsilon_0^{1/2} = Z^{-2/3}.$$



$$\begin{aligned} \mathrm{H}[\mathrm{a}_\lambda] = &-\tfrac{1}{2}\textstyle\sum_{i,j} J_{ij}\tilde{m}_i\tilde{m}_j - \\ &\tfrac{1}{4T}\textstyle\sum_{i,j} J_{ij}^2(1-\tilde{m}_i^2)(1-\tilde{m}_j^2) + \\ &\mathrm{T}\textstyle\sum_i \left(\tfrac{1}{2}\tilde{m}_i^2 + \tfrac{1}{12}\tilde{m}_i^4 + \tfrac{1}{30}\tilde{m}_i^6\right) \\ &- \tfrac{T^2}{18}\textstyle\sum_{i,j} : \tilde{m}_i^3 : g(i,j) : \tilde{m}_j^3 : \end{aligned}$$

$$g(i,j) = \sum_{\epsilon_\alpha < -\epsilon_0} \frac{\psi_\alpha(i)\psi_\alpha(j)}{2 - E_\alpha + \tau^2}$$

The notation $:\tilde{m}_i^2\tilde{m}_i^a:$ means the "normally ordered product", a result of the orthogonality between $\psi_\lambda(i)$ and $g(i,j)$

$$: m_i^3 : = m_i^3 - 3\, m_i \Sigma_\lambda \psi^2_\lambda(i) \langle a^2_\lambda \rangle$$

Hierarchy of Interactions and Superparamagnetic Behaviour

We start from the paramagnetic region $1 \gg \tau \gg \tau_0 = Z^{-2/3}$

$$H_2\{a_\lambda\} = \frac{1}{2}\sum_\lambda (\tau^2 - \epsilon_\lambda)a_\lambda^2 \qquad \text{where } \epsilon_\lambda = E_\lambda - 2.$$

at any τ some modes with $\epsilon_\lambda > \tau^2$ are present that are unstable in the linear approximation.

At $\tau \gg \tau_0$ the density of these modes is very small; moreover, they are well localized so that intermode interactions originating from nonlinear terms in F[m] can be neglected, whereas intramode nonlinearity should be taken into account

$$\rho(\epsilon) \sim \exp\left[-c_1\left(\frac{\epsilon}{\epsilon_0}\right)^{3/4}\right]$$

$$l(\epsilon) \sim Z^{1/3}\epsilon^{-1/4}.$$

Local “clusters”

$$F_\lambda(a_\lambda) = \frac{1}{2}(\tau^2 - \epsilon_\lambda)a_\lambda^2 + \frac{\tau}{6}a_\lambda^4\sum_i \psi_\lambda^4(i) + \frac{1}{90}a_\lambda^6\sum_i \psi_\lambda^6(i)$$

At lower temperatures ($\tau \approx \tau_0$) the number of modes $\tilde{Z}$ interacting strongly with a given condensed mode increases and becomes of order unity ($\tilde{Z} \sim \rho(\epsilon_0)\epsilon_0 l^3(\epsilon_0) = O(1)$

The crucial feature of the large-Z spin-glass model is that the interaction terms can be divided into two types whose magnitudes are strongly different. The terms of the first type depend on the absolute values of all the amplitudes a_λ only; for example

$$a_\lambda^2 a_\mu^2 \sum_i \psi_\lambda^2(i)\psi_\mu^2(i)$$

Only these terms appear to be relevant at $\tau \sim \tau_0$. The terms of the second type depend on the relative signs of the amplitudes, for example

$$a_\lambda \cdot a_\mu a_\nu^2 \sum_i \psi_\lambda(i)\psi_\mu(i)\psi_\nu^2(i).$$

These terms are small in comparison with (4.2.4) owing to the random oscillating nature of the eigenfunctions $\psi_\lambda(i)$ ... $N^{1/2}_{\lambda,\mu,\nu}$ times smaller

For modes with $\epsilon_{\lambda,\mu,\nu} \approx \epsilon_0$, we get $N_{\lambda,\mu,\nu} \sim l^3(\epsilon_0) \sim Z^2 \gg 1$

At $\tau \sim \tau_0$ interactions of the second type are Z times smaller

How it looks for usual phase transition

$$\frac{1}{2}(\tau^2-\epsilon_\lambda)a_\lambda^2+\frac{\tau}{6}u_\lambda^4\sum_i\psi_\lambda^4(i)$$

1

$$F[a_\lambda] = (1/2)\, a^2_\lambda\, (\tau - \varepsilon_\lambda) + a^4 \sum_i \psi_\lambda^4(i)$$

Macroscopic condensation occurs when τ reaches mobility edge position ϵ_c

2

All types of intermode coupling are of the same order of magnitude

Super-paramagnetic behavior at lower temperatures, $|\tau| >> \tau_0$

We neglect all terms with odd powers of a_λ amplitudes and obtain

$$H_0 = \sum_\lambda \left\{ \frac{1}{2}(\tau^2 - \epsilon_\lambda) a_\lambda^2 + \frac{1}{2} a_\lambda^4 \sum_i \psi_\lambda^4(i) \left[\frac{1}{3}\tau + \sum_{\mu\neq\lambda} a_\mu^2 \psi_\mu^2(i) \right] \right\} + \frac{1}{2} \sum_{\lambda\neq\mu} a_\lambda^2 a_\mu^2 \sum_i \psi_\lambda^2(i)\psi_\mu^2(i)$$

$$+ \frac{1}{6} \sum_{\lambda\neq\mu\neq\nu} a_\lambda^2 a_\mu^2 a_\nu^2 \sum_i \psi_\lambda^2(i)\psi_\mu{}^2(i)\psi_\nu^2(i).$$

We used

$$\sum_i \psi_\lambda^4(i) \approx 3 \sum_{i,j} \psi_\lambda^2(i)\psi_\nu^2(j) J_{ij}^2,$$

valid for $Z >> 1$

intermode interaction terms change the coefficients of the terms quadratic and quartic in a_λ in the effective single-mode energy $F_\lambda(a_\lambda)$. To study the influence of mode–mode coupling, we introduce the molecular field B_i,

$$B_i = \sum_\lambda \langle a_\lambda^2 \rangle \psi_\lambda^2(i).$$

At $|\tau| >> \tau_0$ fluctuations of B_i are small $B = \frac{1}{N}\sum_i B_i = \frac{1}{N}\sum_\lambda \langle a_\lambda^2 \rangle$ →

$$\frac{\partial H_0}{\partial a_\lambda^2} = \frac{1}{2}[(\tau + B)^2 - \epsilon_\lambda] + \left(\frac{1}{3}\tau + B\right) a_\lambda^2 \sum_i \psi_\lambda^4(i) = 0$$

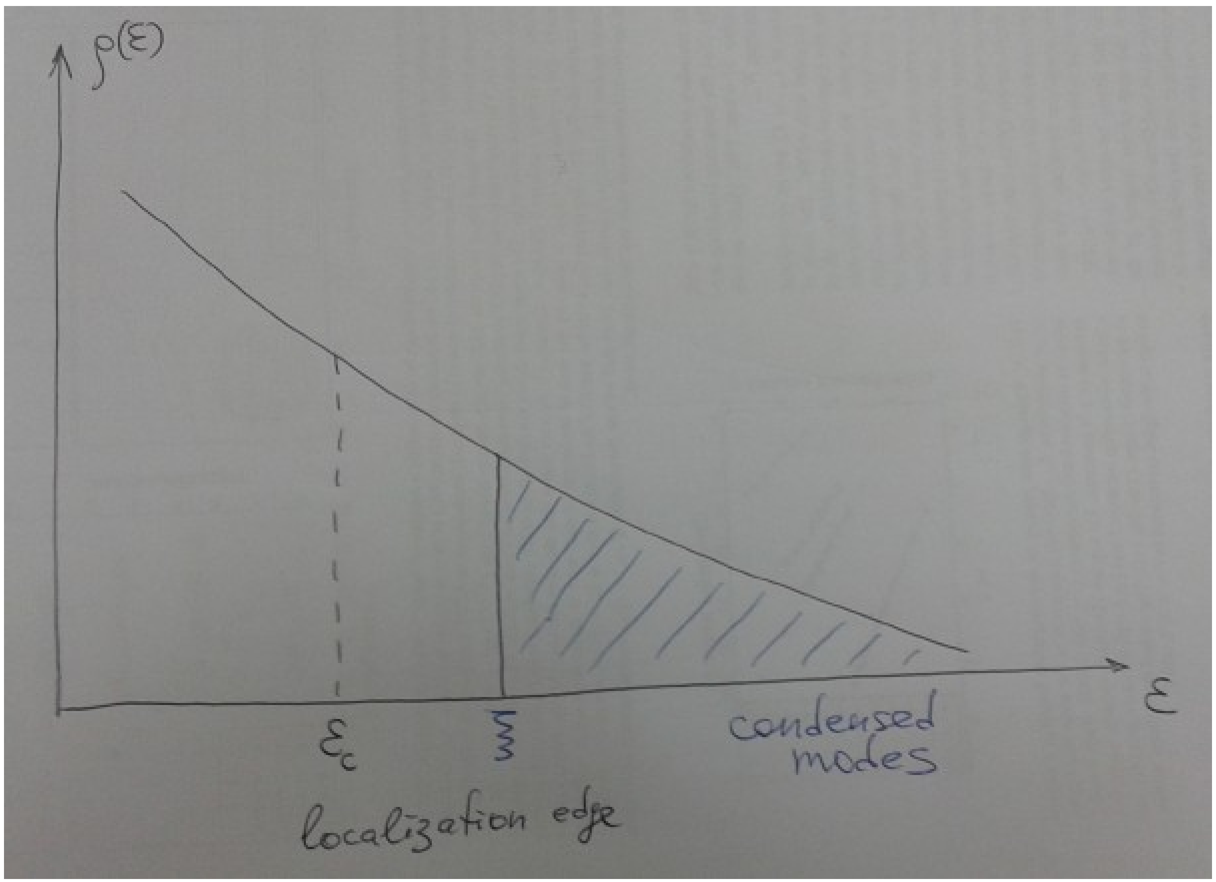
$$a_\lambda^2 = \frac{3}{4|\tau|}(\epsilon_\lambda - \xi)V_\lambda \Theta(\epsilon_\lambda - \xi)$$

here $\xi = (\tau + B)^2$

$V_\lambda = [\sum_i \psi_\lambda^4(i)]^{-1}$ - localization volume

At $|\tau| >> \tau_0$

Self-consistency eq.



$$\tau^2 = \frac{1}{N}\frac{3}{4}\sum_\lambda (\epsilon_\lambda - \xi)V_\lambda \Theta(\epsilon_\lambda - \xi)$$

$$\approx \frac{3}{4}\int \rho(\epsilon)(\epsilon - \xi)V(\epsilon)\, d\epsilon$$

$$V(\epsilon) \sim Z^2 \left(\frac{\epsilon - \epsilon_c}{\epsilon_0}\right)^{-\theta}.$$

For $\theta > 2$

$$\left(\frac{\tau}{\tau_0}\right)^2 \sim \zeta^{2-\theta}, \quad \zeta = \frac{\xi - \epsilon_c}{\epsilon_0}.$$

ζ decreases and tends to approach zero upon T decrease

$\theta = \nu D_2$

Current knowledge of the exponents:

Finite-size scaling and multifractality at the Anderson transition for the three Wigner-Dyson symmetry classes in three dimensions

L up to 100

PHYSICAL REVIEW B 91, 184206 (2015) László Ujfalusi* and Imre Varga†

Orthogonal ensemble:

$D_2 = 1.231$ $\nu = 1.595$ → $\theta = 1.96$

J. Lindinger and A. Rodríguez Phys. Rev. B 96 134202 (2017) Unitary ensemble

A word of caution: we need to average $V_\lambda = [\sum_i \psi_\lambda^4(i)]^{-1}$

instead of the usual IPR $\sum_i \psi_\lambda^4(i)$

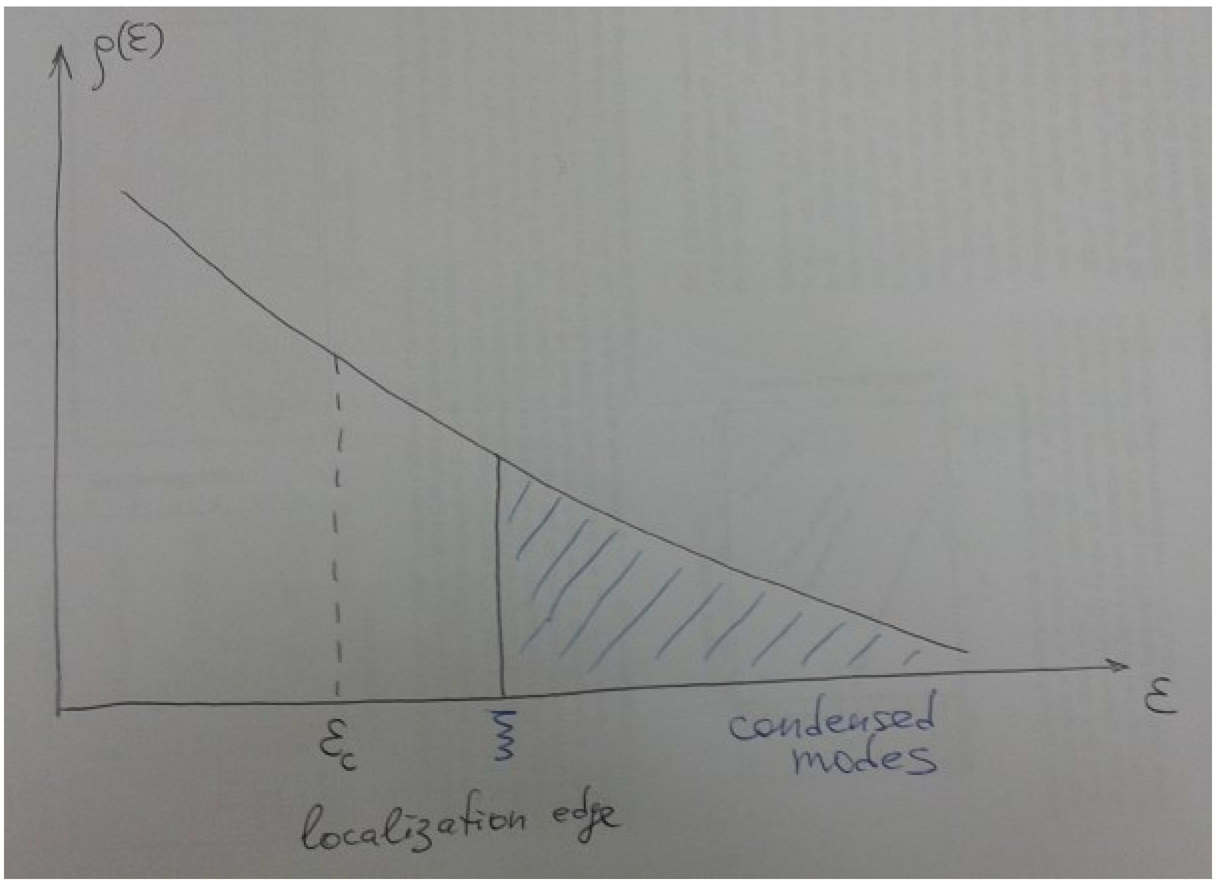
which is used to define D_2

V_λ is determined by the most extended states with small IPR

Thus we need the value of D_2 defined by typical IPR rather than average ones

It leads (Cuevas, Kravtsov 2007) to slightly larger value of D_2

Assuming $\theta \approx 2$ we find a similar solution for self-consistency equation:



$$\left(\frac{\tau}{\tau_0}\right)^2 \sim \ln(1/\zeta)$$

$$\zeta = \frac{\xi - \epsilon_c}{\epsilon_0}.$$

The borderline between condensed and uncondensed modes asymptotically approaches mobility edge

Effective Superspin Interaction and Hierarchy of Superparamagnets

Now we should include "odd" terms in the Hamiltonian of slow modes
These terms are smaller than "even" ones, but they also grow
with T and ζ decrease and eventually become relevant
at some temperature when $T_0 - T = -\tau = \tau_1$

τ_1 is determined by the condition that effective interaction
between signs of amplitudes a_λ i.e. $\sigma_\lambda = a_\lambda / |a_\lambda|$
start to interact strongly

$$H_1\{\sigma_\lambda\} = -\frac{1}{2}\sum_{\lambda,\mu} I_{\lambda\mu}\sigma_\lambda \cdot \sigma_\mu .$$

At leading order in τ

$$\delta B_i \equiv B_i - B$$

$$I_{\lambda\mu} = |a_\lambda a_\mu| \sum_i \psi_\lambda(i)\psi_\mu(i)\delta B(i)$$

Signs of $I_{\lambda\mu}$ are random i.e. this interaction is also of SG-type

Interaction of "superspins": major parameters

$$I_{\lambda\mu} = |a_\lambda a_\mu| \sum_i \psi_\lambda(i)\psi_\mu(i)\delta B(i)$$

The couplings $I_{\lambda\mu}$ are random and weakly (at most) correlated owing to the random character of the eigenfunctions $\psi_\lambda(i)$ and the "background" field δB_i.

The effective strength of the σ_λ interaction is characterized by the parameter

(we put it = 1)

$J_1^2 = \Sigma_\lambda I^2_{\lambda\mu} \sim \tilde{Z} I^2_{\lambda\mu}$ That is analog of $J_0^2 = \Sigma_j J^2_{ij}$

The main contribution comes from the largest-scale modes with $\epsilon_\lambda - \xi \approx \xi - \epsilon_c = \zeta\epsilon_0$

The corresponding eigenfunctions $\psi_\lambda(i)$ overlap strongly, with mean coordination number $\tilde{Z}$ >> 1

$$\tilde{Z}(\zeta) \sim \zeta^{-\kappa}$$

$$J_1(\zeta) \sim Z^{-1/3}\zeta^{-\phi}$$

$$I_{\lambda\mu} = |a_\lambda a_\mu| \sum_i \psi_\lambda(i)\psi_\mu(i)\delta B(i)$$ $\delta B_i \equiv B_i - B$

Estimate the values of $I^{(1)}$ and $I^{(2)}$ at $\tau \sim \tau_0$

$$a_\lambda^2 \sim \frac{\epsilon_\lambda - \tau_0^2}{|\tau_0|} V(\epsilon_\lambda \sim \epsilon_0) \sim Z^{4/3} \qquad V(\epsilon_\lambda \sim \epsilon_0) \sim Z^2$$

$$\delta B_i \sim \tau_0 \qquad \delta B(i,j) \sim \tau_0 \qquad \psi_\lambda(i)\psi_\mu(j) \sim \frac{1}{V_{\lambda,\mu}}$$

$I_{\lambda\mu}$ contains sign-alternating sum over ~ Z^2 sites $\rightarrow$ Z factor

Combining all factors we find for $\tau \sim \tau_0$ $\quad I^{(1)} \sim Z^{-1/3}$

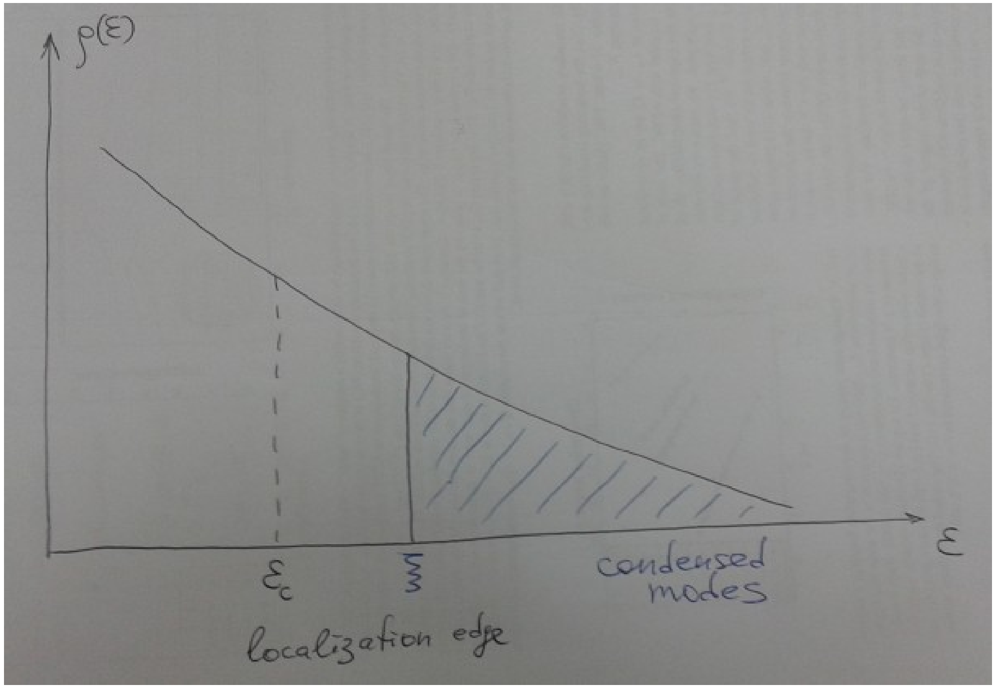
At lower T's the sum grows as power of ζ : $\quad J_1(\zeta) \sim Z^{-1/3}\zeta^{-\phi}$

Assume by now that the first term $I^{(1)}$ is the major one (to be checked!)

Effective coupling strength J_1 becomes of the order of unity at

$$\zeta = \zeta_1 \sim Z^{-1/3\phi}$$

Effective coordination number at $\zeta = \zeta_1$ is $\boxed{\tilde{Z} \sim Z^{\kappa/3\phi}}$



$$\zeta = \frac{\xi - \epsilon_c}{\epsilon_0}. \qquad \ln\frac{1}{\zeta} \sim (\tau/\tau_0)^2$$

$$\tau_1 \sim \tau_0 \sqrt{\frac{\ln Z}{\phi}} > \tau_0$$

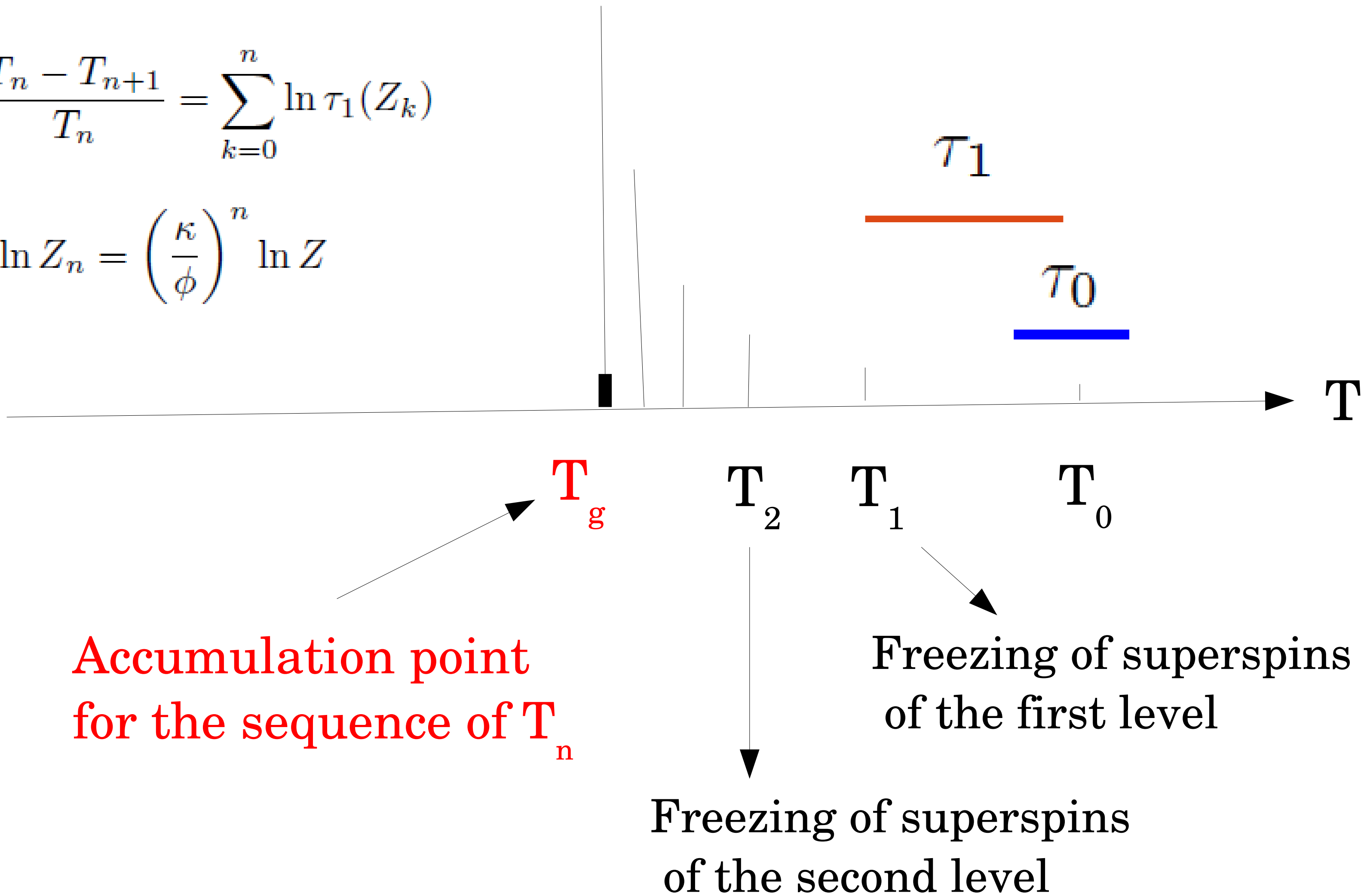
The key point: is the ratio
$\kappa/3\phi$
larger than unity ?

If $\kappa/3\phi > 1$ then $\boxed{\tilde{Z} > Z}$ → New SG problem is the same type as original one

Sequence of fractal clusters that grow upon temperature decrease

$$\ln \frac{T_n - T_{n+1}}{T_n} = \sum_{k=0}^{n} \ln \tau_1(Z_k)$$

$$\ln Z_n = \left(\frac{\kappa}{\phi}\right)^n \ln Z$$



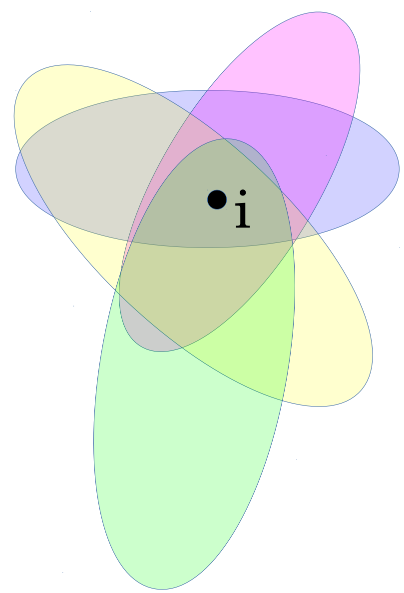
Local magnetization is a function of "super-spins" that belong to overlapping fractal clusters

$$m_i = \sum_\lambda a_\lambda \psi_\lambda(i)$$

$$a_\lambda = \sum_\alpha a^{(2)}_\alpha \psi^{(2)}_\alpha(\lambda)$$

$$a^{(2)}_\alpha = \sum_\beta a^{(3)}_\beta \psi^{(3)}_\beta(\alpha)$$

… .. .. … ...



With T decrease, the system inevitably goes out of equilibrium as barriers for largest condensed modes grow above $T \ln(t_{obs}/t_0)$

Conclusions

- Spin glass with large Z may present very special phase transition scenario with hierarchy of fractal clusters, instead of usually presumed scaling behavior.
- To prove (or disprove) such a scenario, a careful analysis of wave-function statistics near 3D AT is needed.
- The same approach can be used for Gauge Glasses (randomly frustrated Josephson networks)
- Generalization for the Quantum Spin Glass problem

  seems to be possible

Alternative scenario

If $\kappa/3\phi < 1$ then $\tilde{Z} < Z$

In this case effective coordination number drops during each step of RG transformations and critical behavior is the same as in the Spin glass model with $Z \sim 1$

The values of the exponents κ and ϕ are to be determined by the statistics of critical eigenfunctions near Anderson transition

Few problems to solve

- Perform numerical analysis of the wave-function statistics for 3D real Anderson model in order to determine relevant critical exponents

  θ, κ, ϕ and to find which scenario is realized for Ising spin-glass transition: usual scaling v/s critical hierarchy ?
- The same problem for 3D unitary class with application to the gauge glass transition
- Compare XY spin glass with real random $\mathbf{J}_{ij}$ and

  gauge glass with complex random $\mathbf{J}_{ij}$