# One-Dimensional Spin Glass with Oscillating Long-Range Interaction

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and Chapter 3.2 in

**SPIN GLASSES AND RELATED PROBLEMS**

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$$V(\mathbf{p}) = \frac{V_0}{\left(\frac{p^2 - Q^2}{2Q\kappa}\right)^2 + 1}; \qquad \kappa \ll Q.$$

$$V_1(x) = \kappa V_0 e^{-\kappa|x|} \cos Q x$$

$$V_3(\mathbf{x}) = \frac{Q}{2\pi} \kappa V_0 e^{-\kappa x} \frac{\sin Q x}{x}$$

$$\kappa \ll c \ll Q.$$

chain of Ising spins $\sigma_i = \pm 1$ with the oscillating interaction:

$$H = \frac{1}{2} \sum_{ij} \sigma_i V_{ij} \sigma_j - \sum_i \sigma_i h_i$$

$$V_{ij} = \kappa e^{-\kappa|x_i - x_j|} \cos Q(x_i - x_j). \qquad (1)$$

$$H=\frac{1}{2}\sum_{ij}\sigma_i V_{ij}\sigma_j-\sum_i \sigma_i h_i$$

$$V_{ij}=\kappa\, e^{-\kappa|x_i-x_j|}\cos Q(x_i-x_j).$$

$$Z=\sum_{\{\sigma_i=\pm 1\}}\int \mathscr{D}\psi\,\mathscr{D}\psi^*\exp\left\{-\frac{1}{T}\int\left[\frac{1}{\kappa^2}\left|\left(i\frac{d}{dx}+Q\right)\psi\right|^2\right.\right.$$

$$\left.\left.+|\psi|^2\right]dx+\frac{1}{T}\sum_i \sigma_i(\psi(x_i)+\psi^*(x_i)+h)\right\}$$

$$=\int \mathscr{D}\psi\,\mathscr{D}\psi^*\exp\left\{-\frac{1}{T}H_{\mathrm{eff}}[\psi]\right\} \tag{2}$$

where

$$H_{\mathrm{eff}}[\psi]=\int\left[\kappa^{-2}\left|\left(i\frac{d}{dx}+Q\right)\psi\right|^2+|\psi|^2\right.$$

$$\left.-T\sum_i\delta(x-x_j)\ln\mathrm{ch}\frac{\psi+\psi^*+h}{T}\right]dx. \tag{3}$$

Expand in powers of $\psi$

$$H_{\mathrm{eff}}[\psi] = \int \left\{ \kappa^{-2} \left| \left( i \frac{d}{dx} + Q \right) \psi \right|^2 \right.$$

$$+ \left( 1 - \frac{1}{T} \sum_j \delta(x - x_j) \right) |\psi|^2$$

$$\left. + \sum_j \delta(x - x_j) \left[ -\frac{1}{2T} (\psi^2 + \psi^{*2}) + \frac{1}{2T^3} (\psi + \psi^*)^4 \right] \right\} dx.$$

$$\gamma = \kappa / c \ll 1$$

At $T < T_c = c$ the averaged value of the coefficient at $|\psi|^2$, which is equal to $\tau = 1 - c/T$, becomes negative, which implies the instability of the state with $\langle \psi \rangle = 0$.

$t \ll \; t_m \sim \exp(\gamma^{-1} |\tau|^{3/2})$. Phase slips are absent

At $|\tau| \gg \gamma^{2/3}$ it is convenient to pass over to the new variables: the amplitude and phase of the field $\psi$:

$$\psi = \rho \exp(i\varphi + iQx). \tag{5}$$

The free energy $H_{\text{eff}}[\psi]$ from (3) can be minimized over $\rho$, which leads to the phase-dependent energy

$$H[\varphi] = \int \left\{ \frac{\rho^2}{\kappa^2} \left( \frac{d\varphi}{dx} \right)^2 - T \sum_i \delta(x - x_i) \cdot \left[ \ln \operatorname{ch} \left( \frac{2\rho}{T} \cos(Qx + \varphi) \right) - \left\langle \ln \operatorname{ch} \left( \frac{2\rho}{T} \cos\alpha \right) \right\rangle_\alpha \right] \right\} dx \tag{6}$$

where $\langle F(\alpha) \rangle_\alpha = \int_0^{2\pi} F(\alpha)\, d\alpha / 2\pi$, and $\rho$ is approximately determined by the equation

$$\rho = c \left\langle \cos\alpha \cdot \operatorname{th} \left( \frac{2\rho}{T} \cos_\alpha \right) \right\rangle. \tag{7}$$

Equation (7) is obtained if $H[\varphi]$ is neglected (the validity of it is discussed below) in comparison with

$$F_{\text{MFA}} = H_{\text{eff}}[\psi] - H[\varphi] = \rho^2 - cT \left\langle \ln \operatorname{ch} \frac{2\rho \cos_\alpha}{T} \right\rangle_\alpha. \tag{8}$$

## II. The Vicinity of the Transition Point $\gamma^{2/3} \ll |\tau| \ll 1$

$$F_{\mathrm{MFA}} = \tau \rho^2 + \frac{c h^2}{2 T^3} \rho^4 - \frac{c h^2}{2T} + \frac{c \rho^2}{T^3} h^2. \tag{9}$$

For the specific heat $C(T)$ and magnetic susceptibility $\chi(T)$ we obtain:

$$C(T) = c\,\theta(-\tau) \qquad (|\tau| \gg \gamma^{2/3}) \tag{10}$$

$$\chi(T) = c/T \qquad (\tau \gg \gamma^{2/3}) \tag{11a}$$

$$\chi(T) = 1 - |\tau| \qquad (-\tau \gg \gamma^{2/3}). \tag{11b}$$

Formula (11b) is valid at observation times that are not too large (see below). Formula (10) and (11) are similar to those in the Mattis model [12] of spin glasses without frustrations, which is quite natural, since the averages $\langle \sigma_i \rangle$ are also expressed through slow variables in our model. The difference is that we have two variables, $\rho$ and $\varphi$:

$$\langle \sigma_i \rangle = -\frac{\partial F}{\partial h_i} = \operatorname{th}\left[\frac{2\rho}{T} \cos(Q x_i + \varphi_i)\right]$$

# Phase-dependent Hamiltonian with "pinning"

$$H[\varphi(x), Q] = |\tau| \int dx \left[\gamma^{-2}\left(\frac{d\varphi}{dx}\right)^2 - c\sum_j \delta(x-x_j) \cdot \cos 2(Qx+\varphi)\right].$$

$$\langle \cos(\varphi(x)-\varphi(0))\rangle \sim \exp(-xL_\varphi^{-1}),$$

$$L_\varphi = Ac^{-1}\gamma^{-4/3} \gg c^{-1} \qquad (A\sim 1)$$

Hamiltonian (13) coincides with the one studied in [9–11] in connection with the problem of charge-density wave pinning by impurities.


9. Fukuyama, H., Lee, P.A.: Phys. Rev. **B17**, 535 (1977)
10. Feigel'man, M.V.: Zh. Eksp. Theor. Fiz. **79**, 1095 (1980)
11. Vinokur, V.M., Mineev, M.B., Feigelman, M.V.: Zh. Eksp. Theor. Fiz. **81**, 2142 (1981)


# Stochastic Transfer Matrix method [10]

$$\Psi_N(\varphi_N) = \mathrm{const}\cdot\exp\left[-\frac{\varepsilon_N(\varphi_N)}{T}\right].$$

$$\varepsilon_{N+1}(\varphi)-\varepsilon_N(\varphi) = -\frac{1}{2}\frac{l_{N+1}}{v_F}\left(\frac{\partial\varepsilon_N}{\partial\varphi}\right)^2 + \frac{T}{2}\frac{l_{N+1}}{v_F}\frac{\partial^2\varepsilon_N}{\partial\varphi^2} - V\cos(\varphi+Qx_{N+1}).$$

**a Hamilton-Jacobi equation (with a discrete imaginary "time" $N$) corresponding to the "equation of motion"**

$$v_F\varphi'' + V\delta(x-x_k)\sin(\varphi+Qx) = 0.$$

$$\varepsilon_N(\varphi) = -\beta_N\cos(\varphi-\gamma_N)$$

$$\beta_0 = (cv_FV^2)^{1/3}.$$

$$\frac{\partial}{\partial\beta}\left(\frac{\partial W}{\partial\beta} - \frac{W}{\beta}\right) = 0\,.$$

$$W_0(\beta) = \exp(-\Phi_0(\beta)) \sim \frac{\beta}{\beta_*^2}\ln\left(\frac{\beta_*}{\beta}\right).$$

**Zh. Eksp. Teor. Fiz. 81, 2142–2159** (1981)

# Stochastic Transfer Matrix method

$$\varepsilon_{N+1}(\varphi)-\varepsilon_N(\varphi)=-\frac{1}{2}\frac{l_{N+1}}{v_F}\left(\frac{\partial\varepsilon_N}{\partial\varphi}\right)^2+\frac{T}{2}\frac{l_{N+1}}{v_F}\frac{\partial^2\varepsilon_N}{\partial\varphi^2}-V\cos(\varphi+Qx_{N+1}).$$

$$G^{-1}(x,x)=\frac{\partial^2\varepsilon_x}{\partial\varphi_x^2}\bigg|_{\partial\varepsilon_x/\partial\varphi_x=0}=\beta_x.$$

$$\varepsilon_N(\varphi)=\sum_{n=2}^{\infty}\frac{\beta_N^{(n)}}{n!}(\varphi-\gamma_N)^n+\varepsilon_N(\gamma_N);\quad \beta_N^{(2)}\equiv\beta_N>0$$

$$0=\frac{\partial\varepsilon_{N+1}}{\partial\varphi}\bigg|_{\gamma_{N+1}}=\frac{\partial\varepsilon_N}{\partial\varphi}\bigg|_{\gamma_{N+1}}-\frac{\partial\varepsilon_N}{\partial\varphi}\bigg|_{\gamma_{N+1}}\frac{\partial^2\varepsilon_N}{\partial\varphi^2}\frac{l_{N+1}}{v_F}+V\sin(\gamma_{N+1}+Qx_{N+1})$$

$$\Rightarrow\quad \frac{\partial\varepsilon_N}{\partial\varphi}\bigg|_{\gamma_{N+1}}=(\gamma_{N+1}-\gamma_N)\frac{\partial^2\varepsilon_N}{\partial\varphi^2}=\beta_N(\gamma_{N+1}-\gamma_N).$$

$$\gamma_{N+1}-\gamma_N=-\frac{V}{\beta_N}\sin(\gamma_N+Qx_{N+1}).$$

$$\beta_{N+1}-\beta_N=-\beta_N^2\frac{l_{N+1}}{v_F}+V\cos\left(\gamma_N+Qx_{N+1}-\frac{V}{\beta_N}\sin(\gamma_N+Qx_{N+1})\right)+\beta^{(4)}\frac{V^2}{2\beta_N^2}\sin^2(\gamma_N+Qx_N),\qquad(11a)$$

$$0=-\frac{1}{cv_F}\langle\beta^2\rangle+\frac{V^2}{2}\left\langle\frac{1}{\beta}\right\rangle-\frac{V^2}{4}\left\langle\beta^{(4)}\frac{1}{\beta^2}\right\rangle$$

$$\boxed{\beta_0=(cv_FV^2)^{1/3}.}$$

Basic scale of the pinning >> V

# Asymptotic analysis

$W(\beta)$ at $\beta \gg \beta_0$, $\qquad \frac{\partial}{\partial\beta}\left(\frac{V^2}{4}\frac{\partial W}{\partial\beta}+\frac{1}{cv_F}\beta^2 W\right)=0. \qquad W(\beta)=\text{const}\exp[-{}^4/_3(\beta/\beta_0)^3],$

Statistics of shocks in Burgers turbulence

Small $\beta \ll \beta_0$ (but also $\beta > V$)

Here Fourrier expansion works better: $\qquad \varepsilon_N(\varphi)=-\beta_N\cos(\varphi-\gamma_N).$

$$\frac{\partial}{\partial\beta}\left(\frac{\partial W}{\partial\beta}-\frac{W}{\beta}\right)=0\,. \qquad W_0(\beta)=\exp(-\Phi_0(\beta))\sim\frac{\beta}{\beta_*^2}\ln\left(\frac{\beta_*}{\beta}\right).$$

Stability analysis w.r.t. second harmonics selects solution with the Logarithm

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Zh. Eksp. Teor. Fiz. **81**, 2142–2159 (1981)

# Moderately Low Temperature Range $\kappa \ll T \ll T_c = c$

$$\varepsilon_{n+1}(\varphi)-\varepsilon_n(\varphi)=-\frac{\kappa^2}{4\rho_0^2}l_{n+1}\left(\frac{\partial\varepsilon_{n+1}}{\partial\varphi}\right)^2 + \frac{T}{4}\frac{l_{n+1}}{\rho_0^2}\kappa^2\frac{\partial^2\varepsilon_n}{\partial\varphi^2}+V(\varphi+Qx_n)$$

At $T \ll c$ the phase-pinning potential

$$V(\varphi+Qx)=-T\ln\operatorname{ch}\left[\frac{2\rho}{T}\cos(\varphi+Qx)\right]$$

Cusp singularity at low T

replace $l_n$ by $\langle l_n\rangle = c^{-1}$, (at $T \gg \kappa$).

$$\varepsilon_{n+1}(\varphi)-\varepsilon_n(\varphi)=-\frac{\kappa}{4\rho_0^2 c}\left(\frac{\partial\varepsilon_n}{\partial\varphi}\right)^2+V(\varphi+\alpha_n)$$

## Generating functional

$$P(j)=\int \mathscr{D}\mu\,\mathscr{D}\varepsilon\exp\left\{i\sum_n\int d\varphi\left[\mu_n\left(\varepsilon_{n+1}-\varepsilon_n + \frac{\kappa^2}{4\rho_0^2 c}\left(\frac{\partial\varepsilon_n}{\partial\varphi}\right)^2-V(\varphi+\alpha_n)\right)\right]\right\}$$

Let us pass from integrating over $\mu$ to integrating over $M$, so that $M''_{\varphi\varphi}=\mu$.

$$P(j)=\int\mathscr{D}M\,\mathscr{D}\varepsilon\exp\Big(S-\sum_n\int d\varphi\, j_n\varepsilon_n\Big)$$

$$S=i\sum_n\int d\varphi\left\{M_n\frac{\partial^2}{\partial\varphi^2}\left[\varepsilon_{n+1}-\varepsilon_n+\frac{\kappa^2}{4\rho_0^2 c}\left(\frac{\partial\varepsilon_n}{\partial\varphi}\right)^2 - V(\varphi+\alpha_n)\right]\right\}.$$

Average the functional over $\alpha_n$

$$S = \sum_n \left\{ \int i M_n \frac{\partial^2}{\partial \varphi^2} \left[ \varepsilon_{n+1} - \varepsilon_n + \frac{\kappa^2}{4\rho_0^2 c} \left( \frac{\partial \varepsilon_n}{\partial \varphi} \right)^2 \right] d\varphi \right.$$

$$\left. + \ln \left[ \int_0^\pi d\varphi_n \exp(2i\rho_0 M_n(\varphi_n)) \right] \right\}$$

$$= \sum_n \left[ \int i M_n \frac{\partial^2}{\partial \varphi^2} \left[ \varepsilon_{n+1} - \varepsilon_n + \frac{\kappa^2}{4\rho_0^2 c} \left( \frac{\partial \varepsilon_n}{\partial \varphi} \right)^2 \right] d\varphi \right.$$

$$\left. - 2\rho_0^2 \int M_n^2(\varphi)\, d\varphi + \underbrace{\sum_{K=3}^{\infty} a_K \rho_0^K \int M_n^K(\varphi)\, d\varphi}_{\text{small}} \right\}$$

$$S_0 = -\int \left\{ \frac{\partial^2}{\partial \varphi^2} \left[ \frac{\partial \varepsilon}{\partial x} + \frac{\kappa^2}{4\rho_0^2} \left( \frac{\partial \varepsilon}{\partial \varphi} \right)^2 \right] \right\}^2 \frac{1}{8\rho_0^2 c} d\varphi\, dx$$

$$T/c \ll \Phi \ll 1;$$

$$E X^{-1} \sim \kappa^2 \rho_0^{-2} E^2 \Phi^{-2}$$
$$E^2 X^{-1} \Phi^{-3} \rho_0^{-2} c^{-1} \sim 1$$

$\longrightarrow$

$$X \sim \gamma^{-4/3} \Phi^{1/3} c^{-1}$$
$$E \sim \gamma^{-2/3} \Phi^{5/3} c$$

# Estimate for the neglected terms

$$\Delta S = \sum_{K=3}^{\infty} \tilde{a}_K \frac{c}{(\rho_0 c)^K} \int d\varphi\, dx \left\{ \frac{\partial^2}{\partial \varphi^2} \left[ \frac{\partial \varepsilon}{\partial x} + \frac{\kappa^2}{4\rho_0^2} \left( \frac{\partial \varepsilon}{\partial \varphi} \right)^2 \right] \right\}^K$$

$$\Delta S / S_0 \sim \left( \frac{\kappa}{T} \right)^{2/3} \ll 1$$

## Correlation length

The total scale of the $\varepsilon(\varphi)$ variation is determined by fluctuations with $\Phi \sim 1$. Its value $E \sim c\gamma^{-2/3}$ is the same as at $T \sim T_c$.

$$\boxed{\begin{aligned} X &\sim \gamma^{-4/3} \Phi^{1/3} c^{-1} \\ E &\sim \gamma^{-2/3} \Phi^{5/3} c \end{aligned}}$$

Let's find the phase correlation length $L_\varphi$. For this purpose we estimate the phase variation by each step ($|\Delta \bar{\varphi}| = |\bar{\varphi}_n - \bar{\varphi}_{n-1}|$ where $\bar{\varphi}_n$ is the thermal average of $\varphi_n$. Then $L_\varphi$ will be determined by the condition:

$$L_\varphi c \langle (\Delta \bar{\varphi})^2 \rangle = 1$$

Since thermal fluctuations are small, $\bar{\varphi}_n$ is determined by the position of the $\varepsilon(\varphi)$ minimum (perhaps, a local one): $\varepsilon_n'(\bar{\varphi}) = 0$. Using the recurrent equation (23) we get

$$\langle (\Delta \bar{\varphi})^2 \rangle \sim \beta^{-2} = \left\langle \left( \frac{\partial^2 \varepsilon}{\partial \varphi^2} \right)^2 \right\rangle^{-1}$$

$$\beta \sim c\gamma^{-2/3} (T/c)^{-1/3}$$

$$L_\varphi \sim c\gamma^{-4/3} (T/c)^{-2/3}$$

$$X_{\min} \sim c^{-1} \gamma^{-4/3} (T/c)^{1/3}, \quad (\Phi \sim T/c)$$

Is much smaller than $L_\varphi$

# Structure of free energy minima: now we show that the estimates

$$X \sim \gamma^{-4/3} \Phi^{1/3} c^{-1}$$

$$E \sim \gamma^{-2/3} \Phi^{5/3} c$$

Indicate the fractal structure of free energy minima

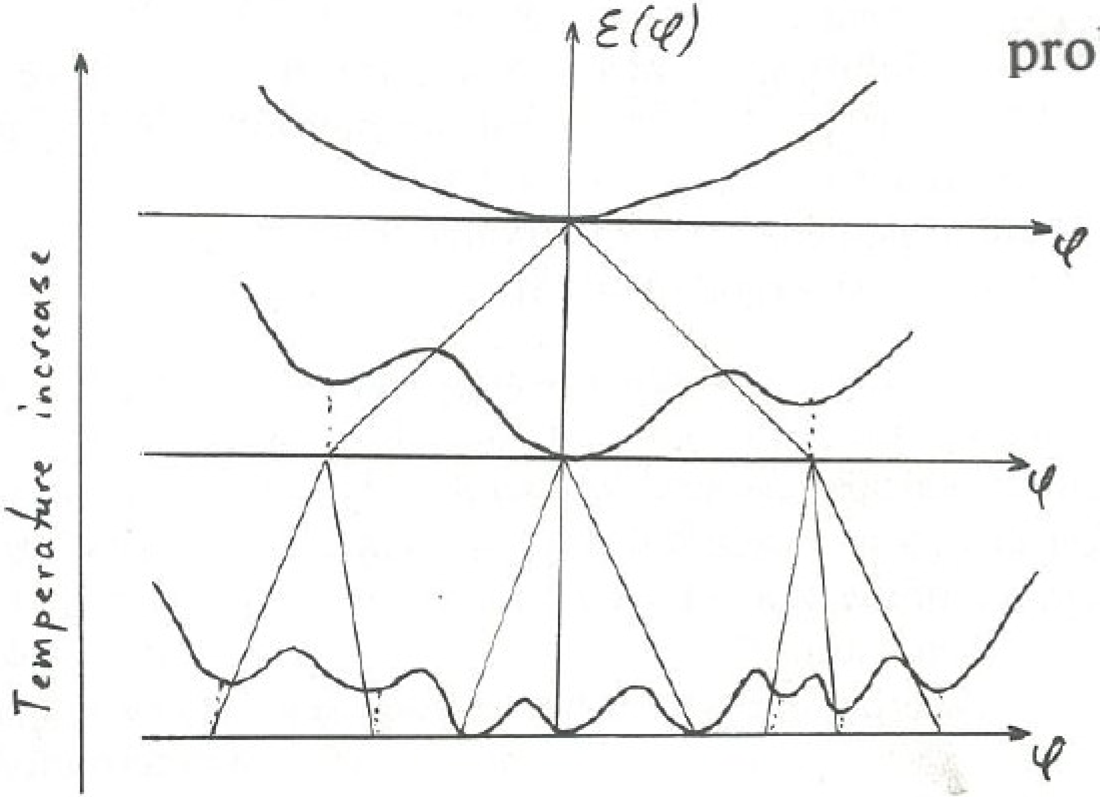


**Figure 11** Hierarchical structure of metastable states.

probability of finding a zero of $f(\phi^{I} + \Phi)$

within an interval of width $\Phi_0$

$$P(\Phi) \sim \frac{\Phi_0 \beta}{F(\Phi)} = \left(\frac{\Phi_0}{\Phi}\right)^{2/3}$$

the total number of zeros

$$M_0 \sim \int_{\Phi_0}^{\pi} \frac{\mathrm{d}\Phi}{\Phi_0} P(\Phi) \sim \left(\frac{\pi}{\Phi_0}\right)^{1/3} \sim T^{-1/3}$$

The minima of $\epsilon(\phi)$ constitute a fractal set with fractal dimension $D_{\mathrm{f}} = \frac{1}{3}$

consider the behaviour of the function $f(\phi) = \partial\epsilon/\partial\phi$ at $\phi = \phi^{I} + \Phi$, where $\phi^{I}$ is any zero of $f(\phi)$. The characteristic scale of $f(\phi^{I} + \Phi)$ is $F(\Phi)$

$$F(\Phi) \sim E(\Phi)/\Phi \sim (\Phi/\gamma)^{2/3}$$

$$X \sim \gamma^{-4/3} \Phi^{1/3} c^{-1}$$

$$E \sim \gamma^{-2/3} \Phi^{5/3} c$$

the scale $\beta$ of $\partial f/\partial\phi$ is determined by the smallest-scale fluctuations with $\Phi \sim \Phi_0$:

$$\beta \sim \gamma^{-2/3} \Phi_0^{-1/3}$$

# Observables

First of all, we find the free energy of the system in a certain metastable state. As was shown at the end of the previous section, the barrier height at $T \gg \kappa$ is much larger than $T$; therefore, when calculating the characteristics of the system corresponding to short time scales (quasiequilibrium I), the phase $\varphi_i$ can be considered constant at a given point $i$ and as satisfying the condition $\varepsilon_i'(\varphi_i)=0$. Thermal fluctuations of $\rho(x)$ can always be neglected; therefore, the free energy $F_I$ coincides with the Hamiltonian $H_{\rm eff}$ for the given configuration of $\{\varphi_i\}$.

$H_{\rm eff}$ is given by Formula (3). For future purposes it is convenient to rewrite it as:

$$F_I = H_{\rm eff} = \int \{\kappa^{-2}[(\nabla\rho)^2 + \rho^2(\nabla\varphi)^2] + \rho^2\}\,dx - T\sum_j \ln\operatorname{ch}\left(\frac{h+2\rho\operatorname{Cos}(\varphi+\alpha_j)}{T}\right) \qquad (33)$$

$$\chi_{\rm eq} = \frac{c}{T}$$

$$\langle\sigma_i\rangle = -\frac{\partial F}{\partial h_i} = \operatorname{th}\left[\frac{2\rho}{T}\cos(Qx_i+\varphi_i)\right]$$

## Distribution function of phases at local minimum in nearly uniform

$$\rho(\varphi_i+\alpha_i) = \frac{1}{2\pi} + O\left(\left(\frac{\kappa}{T}\right)^{2/3}\right)$$

The behavior of the magnetic susceptibility $\chi = -\dfrac{\partial^2 F}{\partial h^2}$ is different at different times of observation.

Let's first consider $\chi$ in the region where $t_0 \ll t \ll t_1$ (for quasiequilibrium I, see the end of the previous section). Inserting Formula (33) for $F$ we obtain:

$$\chi_I = \frac{c}{T}\left\langle \operatorname{ch}^{-2}\left(\frac{2\rho_0\operatorname{Cos}(\varphi_i+\alpha_i)}{T}\right)\right\rangle = \tfrac{1}{2} \qquad (42)$$

# Slowly time-dependent behavior at $t_0 \ll t \ll t_1$

$t_0 \sim \exp(E_0/T) \sim \exp[(T/\gamma)^{2/3}]$ $\qquad$ $t_1 \sim \exp(\gamma^{-2/3}/T)$

One of the major characteristics: dissipative response

$$\langle \sigma_i \rangle \approx \operatorname{sign}[(\cos(\phi_i + Qx_i)]$$

$$\operatorname{Im}\chi(\omega) \sim \frac{1}{T}\int \mathrm{d}E\,\mathrm{d}\Delta\, R(E,\Delta)\frac{\omega\tau(E)}{1+[\omega\tau(E)]^2}N(E)\operatorname{sech}^2\frac{\Delta}{2T}.$$

$\tau(E) \sim \mathrm{e}^{E/T}$ $\quad$ $E$ is the free-energy barrier between two states

$\Delta$ is the energy difference between these states

$R(E, \Delta)$ is the joint probability density for barriers E and asymmetry $\Delta$

$N(E) \sim \Phi(E)X(E)$ is the number of spins that flip in the course of the transition between two metastable states

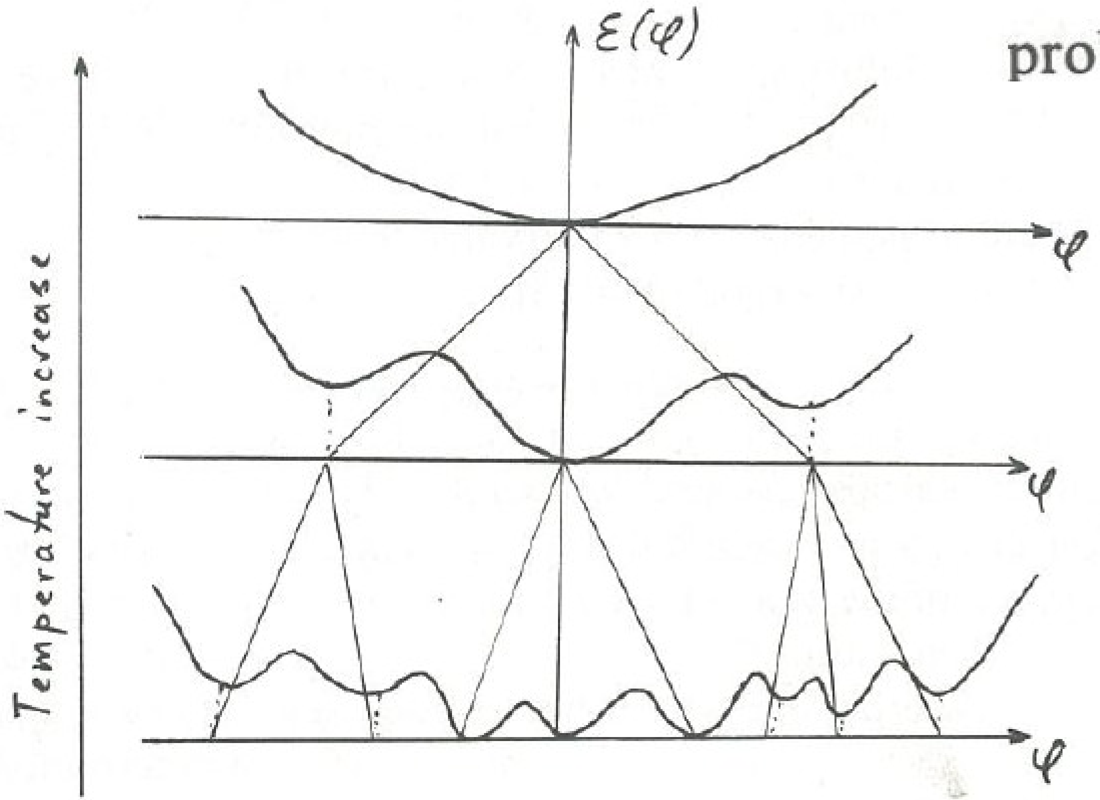


Figure 11 Hierarchical structure of metastable states.

probability of finding a zero of $f(\phi^{I}+\Phi)$ within an interval of width $\Phi_0$

$$P(\Phi) \sim \frac{\Phi_0 \beta}{F(\Phi)} = \left(\frac{\Phi_0}{\Phi}\right)^{2/3}$$

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The minima of $\epsilon(\phi)$ constitute a fractal set with fractal dimension $D_f = \frac{1}{3}$

consider the behaviour of the function $f(\phi) = \partial\epsilon/\partial\phi$ at $\phi = \phi^{I} + \Phi$, where $\phi^{I}$ is any zero of $f(\phi)$. The characteristic scale of $f(\phi^{I}+\Phi)$ is $F(\Phi)$

$$\underline{F(\Phi) \sim E(\Phi)/\Phi \sim (\Phi/\gamma)^{2/3}}$$

$$X \sim \gamma^{-4/3} \Phi^{1/3} c^{-1}$$

$$E \sim \gamma^{-2/3} \Phi^{5/3} c$$

the scale $\beta$ of $\partial f/\partial\phi$ is determined by the smallest-scale fluctuations with $\Phi \sim \Phi_0$:

$$\underline{\beta \sim \gamma^{-2/3} \Phi_0^{-1/3}}$$

$$\mathrm{Im}\,\chi(\omega) \sim \frac{1}{T}\int \mathrm{d}E\,\mathrm{d}\Delta\, R(E,\Delta)\,\frac{\omega\tau(E)}{1+[\omega\tau(E)]^2}\,N(E)\,\mathrm{sech}^2\frac{\Delta}{2T}.$$

$$N(E) \sim \Phi(E)X(E) \sim (E/\gamma)^{4/5}$$

$$M(E) \sim \frac{1}{\gamma^{2/15}E^{1/5}} \qquad X(E) \equiv X(\Phi(E)) \sim \gamma^{-6/5}E^{1/5}$$

Thus the linear density of relaxation modes with barriers in the interval $(E, E+\mathrm{d}E)$ is given by

$$W(E)\,\mathrm{d}E \sim \frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{M(E)}{X(E)}\right)\mathrm{d}E \sim \gamma^{16/15}\frac{\mathrm{d}E}{E^{7/5}}. \qquad (3.2.16)$$

characteristic scale of $\Delta$ is of order $E$, so that $\int \mathrm{d}\Delta\, R(E,\Delta)\,\mathrm{sech}^2\frac{\Delta}{2T} \sim W(E)\,\frac{T}{E}$

Combining all above estimates:

$$\mathrm{Im}\,\chi(\omega) \sim \int \frac{\mathrm{d}E}{E}\,W(E)N(E)\,\frac{\omega\tau(E)}{1+[\omega\tau(E)]^2} \sim \frac{\gamma^{4/15}}{(T\ln\omega^{-1})^{3/5}},$$

In the time domain: $c(t) \sim \dfrac{\gamma^{4/5}}{(T\ln t)^{3/5}}.$

$\chi(t) = -(1/T)\, dC(t)/dt \quad (t>0)$ Fluct-diss. relation

# What did we learned ?

- 1D Ising spin glass with long-range interaction can be described by slow phase variable
- Free energy relief in terms of this variable has fractal structure whose parameters can be estimated
- Upon temperature decrease, more and more fine tree-like structure develops
- Relaxation is logarithmic in a broad range

  of time-scales

# 5 problems to solve

1. Generating functional (defined in slide 7) and corresponding action

$$S_0 = -\int \left\{ \frac{\partial^2}{\partial\varphi^2} \left[ \frac{\partial\varepsilon}{\partial x} + \frac{\kappa^2}{4\rho_0^2} \left( \frac{\partial\varepsilon}{\partial\varphi} \right)^2 \right] \right\}^2 \frac{1}{8\rho_0^2 c} \, d\varphi \, dx$$

refer to the free energy $\varepsilon^{>}_{n}(\varphi)$ defined for the recursion > from the left end of the chain to the reference site n

Actually one should consider the sum $\varepsilon_n(\varphi) = \varepsilon^{>}_{n}(\varphi) + \varepsilon^{<}_{n}(\varphi)$ of two parts of free energy since this is the physical free energy in the middle of the chain

**The problem:**

a) to produce scaling estimates similar to those presented in slides 10-11 , but the total $\varepsilon_n(\varphi)$ ;

b) to check if the distribution of zeros of the function $d\varepsilon_n(\varphi)/d\varphi$ is the same as was derived for $d\varepsilon^{>}_{n}(\varphi)/d\varphi$ (slide 13) ; if it is not the same, to derive the correct one.

2. Imaginary part of response Im $\chi(\omega)$ was calculated (slide 17) assuming Gibbs distribution for different metastable states, each of them defined by some minimum of function $\varepsilon_n(\varphi)$. However, this assumption of full equilibrium is not valid if aging dynamics on timescales $t_a$ is considered Namely, thermodynamic Gibbs distribution will be established for modes separated by energy barriers $E << T \ln (t_a/t_0)$ , whereas modes with barriers $E >> T \ln (t_a/t_0)$ will be populated just randomly. As a result, function $1/\cosh^2(\Delta/2T)$ in the integral on slide 17 should be replaced by some non-equlibrium function, dependent on the value of $t_a$

**The problem:** to find (approximately) this non-equilibrium and non-stationary distribution function for the range of $E \sim T \ln (t_a/t_0)$ and then calculate aging part of the response function Im $\chi(\omega)$ .

3. Simulate numerically this 1D spin glass model (for example, via Monte-Carlo method); check if the fractal structure of low-free-energy minima does actually exist; find overlap distributions and other characteristics. First step: to choose relevant range of the parameters $(Q,c,\kappa)$, number of spins N, temperature T
4. "Intermediate" numerical approach: try to simulate random functions $\varepsilon(\varphi,x)$ with probability distribution defined by the action $S_0$ (slide 19) and check the estimates I proposed
5. Consider quantum version of the same Ising problem: add to the Hamiltonian a transverse-field term $\Gamma \Sigma_i \sigma_i^x$ with relatively small $\Gamma \ll T_c$ and try to derive an effective action in terms of the phase variable $\varphi$. What happens then to the fractal structure of energy valleys described on slide 13 ? Will quantum parameter $\Gamma$ provide some low cutoff effective in the T=0 limit ?