

(Ising SG to be discussed here)

Critical hierarchy of modes in 3D spin glasses

Lecture based on L.Ioffe & M.Feigel'man results (1985-1990) + new analysis

Hierarchical structure of an Edwards-Anderson spin glass

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SPIN GLASSES AND RELATED PROBLEMS

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1. Thouless-Anderson-Palmer method for a finite-range Spin Glass: selection of slow modes and eigenfunctions of the J_{ij} matrix
2. Effective Hamiltonian of slow modes: parity-based classification
3. Formation of super-paramagnetic clusters and their interactions
4. Discrete RG transformation and critical hierarchy

TAP method for a finite-range Spin Glass: selection of slow modes and eigenfunctions of the J_{ij} matrix

D.Thouless, P.Anderson & R.Palmer 1977:

$$F = -\frac{1}{2} \sum_{i,j} J_{ij} m_i m_j - \frac{1}{4T} \sum_{i,j} J_{ij}^2 (1 - m_i^2)(1 - m_j^2) + \frac{T}{2} \sum_i \left[(1 + m_i) \ln \frac{1 + m_i}{2} + (1 - m_i) \ln \frac{1 - m_i}{2} \right]$$

Near T_c expand F in powers of m_i then use basis of eigenfunctions of J_{ij}

$$m_i = \sum_{\lambda} a_{\lambda} \psi_{\lambda}(i), \quad \sum_j J_{ij} \psi_{\lambda}(j) = E_{\lambda} \psi_{\lambda}(i).$$

For

$$Z = N \rightarrow \infty \quad \rho(E) = (2\pi)^{-1} (4 - E^2)^{1/2} \theta(4 - E^2), \quad m_i = a_0 \psi_0(i) + \delta m_i = a_0 \psi_0(i) + \sum_{\alpha \neq 0} a_{\alpha} \psi_{\alpha}(i),$$

$$F = \frac{1}{2} \tau^2 q + \frac{1}{2} \tau q^2 + \frac{1}{2} q^3 + \frac{1}{3} \sum_i a_0^3 \psi_0^3(i) \left[\sum_{\alpha \neq 0} a_{\alpha} \psi_{\alpha}(i) \right] + \frac{1}{2} \sum_{\alpha \neq 0} (\tau^2 + 2 - E_{\alpha}) a_{\alpha}^2,$$

$$\tau = T - T_0 = T - 1 \text{ and } q = a_0^2 / N$$

$$F\{a_0\} = \frac{1}{2} \tau^2 q + \frac{1}{2} \tau q^2 + \frac{1}{2} q^3 - \frac{1}{18} \sum_{i,j} a_0^3 \psi_0^3(i) g(i, j) a_0^3 \psi_0^3(j), \quad g_{ij} = \delta_{ij} \int \frac{\rho(E) dE}{2 - E + \tau^2} = \delta_{ij}$$

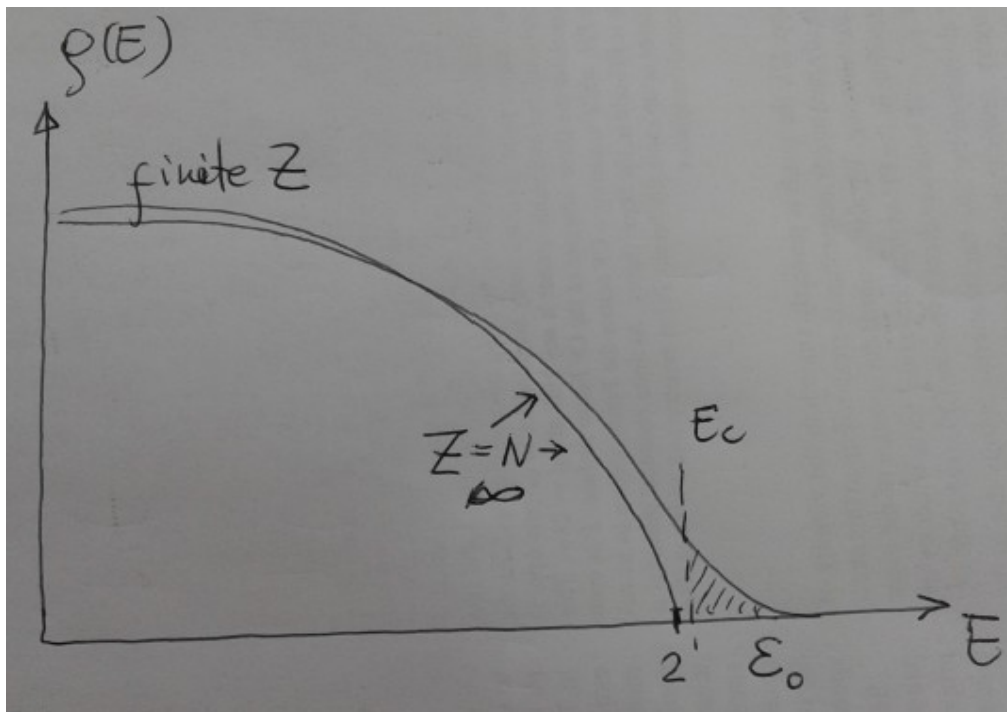
$$F\{a_0\} = \frac{1}{2} \tau^2 q + \frac{1}{2} \tau q^2 + \frac{1}{2} q^3 - \frac{1}{18} \sum_{i,j} a_0^3 \psi_0^3(i) g(i,j) a_0^3 \psi_0^3(j),$$

$$g_{ij} = \delta_{ij} \int \frac{\rho(E) dE}{2 - E + \tau^2} = \delta_{ij} + O(\tau), \quad \sum_j g_{ij} \psi_0(j) = 0 \quad \text{Orthogonality}$$

TAP expression for the free energy: $F\{a_0\} = \frac{1}{6} (\tau + q)^3 - \frac{1}{6} \tau^3, \quad q = \frac{a_0^2}{N} = |\tau|$

Generalization for finite large Z (coordination number)

$$\overline{J_{ij}^2} = cK(r_i - r_j) \quad \int d^3r K(r) = J_0^2 \quad Z = cK^{-3/2} [\int K(r) r^2 d^3r]^{3/2} \gg 1$$



Below $c = J_0 = 1$

$$\epsilon_0 \sim Z^{-4/3}$$

E_c is the mobility edge

$$\rho(E) \sim \exp[-a_1(\epsilon/\epsilon_0)^{3/4}].$$

Tail of localized states

3D finite-range hopping: analysis of the spectrum

$$Z = cK^{-3/2}[\int K(r)r^2 d^3r]^{3/2} \gg 1$$

$$\overline{J_{ij}^2} = cK(r_i - r_j) \quad \int d^3r K(r) = J_0^2 \quad c = J_0 = 1$$

$$\left. \begin{aligned} \rho(E) &= -\frac{1}{\pi} \operatorname{Im} \lim_{l \rightarrow 0} \int \mathcal{D} Q^{\alpha\beta} [(E + i\delta)\delta_{\alpha\beta} + Q^{\alpha\beta}]^{-1} e^{S\{Q\}}, \\ S\{Q\} &= -\frac{1}{4} \sum_{i,j,\alpha,\beta} Q_i^{\alpha\beta} K^{-1}(r_i - r_j) Q_j^{\alpha\beta} \\ &\quad - \frac{1}{2} \operatorname{Tr} \ln [(E + i\delta)\delta_{\alpha\beta} + Q_i^{\alpha\beta}]. \end{aligned} \right\} (4.A.4)$$

$K^{-1}(r_i - r_j)$ can be replaced by $\left(1 - \frac{\nabla^2}{\kappa^2}\right) \delta(r_i - r_j)$

saddlepoint approximation $\mathbf{Q} = -[\mathbf{Q} + \mathbf{1}(E + i\delta)]^{-1}$, the diagonal solution: $Q^{\alpha\beta} = \delta^{\alpha\beta} Q_0$, with

$$Q_0 = \begin{cases} -\frac{1}{2}E + i(1 - \frac{1}{2}E^2)^{1/2} & (|E| \leq 2) \\ -\frac{1}{2}E + \operatorname{sgn}(E) (\frac{1}{4}E^2 - 1)^{1/2} & (|E| \geq 2) \end{cases} \quad \mathbf{Q} = Q_0 \mathbf{1} + \tilde{\mathbf{Q}}.$$

$$S = \operatorname{Tr} \int \left\{ \tilde{\mathbf{Q}} \left[i(-\epsilon)^{1/2} - \frac{1}{2\kappa^2} \nabla^2 \right] \tilde{\mathbf{Q}} + \frac{1}{6} \tilde{\mathbf{Q}}^3 \right\} d^3x, \quad \delta G = G(\delta\Sigma)G, \quad \delta\Sigma = \frac{\kappa^3}{2\pi} e^{i\pi/4} (-4\epsilon)^{-1/3}$$

the corrections become important at

$$-\epsilon \leq \epsilon_0 = \frac{\kappa^4}{(4\pi)^{4/3}} \approx Z^{-4/3}$$

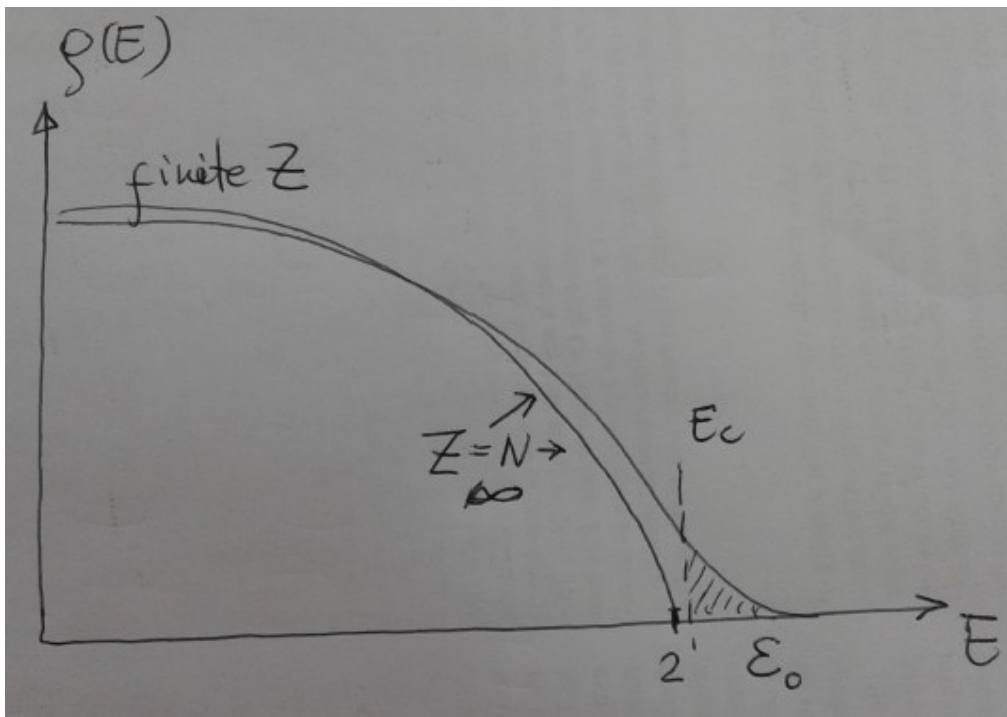
Expansion over eigen-modes for finite large Z

$$m_i = \sum_{\lambda} a_{\lambda} \psi_{\lambda}(i) + \delta m_i = \sum_{\lambda} a_{\lambda} \psi_{\lambda}(i) + \sum_{\alpha} a_{\alpha} \psi_{\alpha}(i)$$

$$\tilde{m}_i = \sum_{\lambda} a_{\lambda} \psi_{\lambda}(i),$$

where we denote “slow” modes by a subscript λ and “fast” ones by α .

$$l(\epsilon) \sim Z^{1/3} \epsilon^{-1/4}, \quad \rho(\epsilon_0) \approx \epsilon_0^{1/2} = Z^{-2/3}.$$



$$\begin{aligned} H[a_{\lambda}] = & -\frac{1}{2} \sum_{i,j} J_{ij} \tilde{m}_i \tilde{m}_j - \\ & \frac{1}{4T} \sum_{i,j} J_{ij}^2 (1 - \tilde{m}_i^2)(1 - \tilde{m}_j^2) + \\ & T \sum_i \left(\frac{1}{2} \tilde{m}_i^2 + \frac{1}{12} \tilde{m}_i^4 + \frac{1}{30} \tilde{m}_i^6 \right) \\ & - \frac{T^2}{18} \sum_{i,j} : \tilde{m}_i^3 : g(i,j) : \tilde{m}_j^3 : \end{aligned}$$

$$g(i,j) = \sum_{\epsilon_{\alpha} < -\epsilon_0} \frac{\psi_{\alpha}(i) \psi_{\alpha}(j)}{2 - E_{\alpha} + \tau^2}$$

The notation : $\tilde{m}_i^2 \tilde{m}_i^{\alpha}$: means the “normally ordered product”,
a result of the orthogonality between $\psi_{\lambda}(i)$ and $g(i,j)$.

$$: m_i^3 : = m_i^3 - 3 m_i \sum_{\lambda} \psi_{\lambda}^2(i) \langle a_{\lambda}^2 \rangle$$

Hierarchy of Interactions and Superparamagnetic Behaviour

We start from the paramagnetic region $1 \gg \tau \gg \tau_0 = Z^{-2/3}$

$$H_2\{a_\lambda\} = \frac{1}{2} \sum_\lambda (\tau^2 - \epsilon_\lambda) a_\lambda^2 \quad \text{where } \epsilon_\lambda = E_\lambda - 2.$$

at any τ some modes with $\epsilon_\lambda > \tau^2$ are present that are unstable in the linear approximation.

At $\tau \gg \tau_0$ the density of these modes is very small; moreover, they are well localized so that intermode interactions originating

$$\rho(\epsilon) \sim \exp \left[-c_1 \left(\frac{\epsilon}{\epsilon_0} \right)^{3/4} \right]$$

from nonlinear terms in $\mathbf{F}[\mathbf{m}]$ can be neglected, whereas intramode nonlinearity should be taken into account

$$l(\epsilon) \sim Z^{1/3} \epsilon^{-1/4}.$$

Local “clusters”

$$F_\lambda(a_\lambda) = \frac{1}{2} (\tau^2 - \epsilon_\lambda) a_\lambda^2 + \frac{\mathcal{T}}{6} a_\lambda^4 \sum_i \psi_\lambda^4(i) + \frac{1}{90} a_\lambda^6 \sum_i \psi_\lambda^6(i)$$

At lower temperatures ($\tau \approx \tau_0$) the number of modes \tilde{Z} interacting strongly with a given condensed mode increases and becomes of order unity ($\tilde{Z} \sim \rho(\epsilon_0)\epsilon_0 l^3(\epsilon_0) = O(1)$)

The crucial feature of the large- Z spin-glass model is that the interaction terms can be divided into two types whose magnitudes are strongly different. The terms of the first type depend on the absolute values of all the amplitudes a_λ only; for example

$$a_\lambda^2 a_\mu^2 \sum_i \psi_\lambda^2(i) \psi_\mu^2(i)$$

Only these terms appear to be relevant at $\tau \sim \tau_0$. The terms of the second type depend on the relative signs of the amplitudes, for example

$$a_\lambda a_\mu a_\nu^2 \sum_i \psi_\lambda(i) \psi_\mu(i) \psi_\nu^2(i).$$

These terms are small in comparison with (4.2.4) owing to the random oscillating nature of the eigenfunctions $\psi_\lambda(i)$ $N_{\lambda,\mu,\nu}^{1/2}$ times smaller

For modes with $\epsilon_{\lambda,\mu,\nu} \approx \epsilon_0$, we get $N_{\lambda,\mu,\nu} \sim l^3(\epsilon_0) \sim Z^2 \gg 1$

At $\tau \sim \tau_0$ interactions of the second type are Z times smaller

How it looks for usual phase transition

1

$$\frac{1}{2} (\tau^2 - \epsilon_\lambda) a_\lambda^2 + \frac{\mathcal{T}}{6} a_\lambda^4 \sum_i \psi_\lambda^4(i)$$

$$F[a_\lambda] = (1/2) a_\lambda^2 (\tau - \epsilon_\lambda) + a_\lambda^4 \sum_i \psi_\lambda^4(i)$$

Macroscopic condensation occurs when τ reaches mobility edge position ϵ_c

2

All types of intermode coupling are of the same order of magnitude

Super-paramagnetic behavior at lower temperatures, $|\tau| \gg \tau_0$

We neglect all terms with odd powers of a_λ amplitudes and obtain

$$H_0 = \sum_{\lambda} \left\{ \frac{1}{2} (\tau^2 - \epsilon_{\lambda}) a_{\lambda}^2 + \frac{1}{2} a_{\lambda}^4 \sum_i \psi_{\lambda}^4(i) \left[\frac{1}{3} \tau + \sum_{\mu \neq \lambda} a_{\mu}^2 \psi_{\mu}^2(i) \right] \right\} + \frac{1}{2} \sum_{\lambda \neq \mu} a_{\lambda}^2 a_{\mu}^2 \sum_i \psi_{\lambda}^2(i) \psi_{\mu}^2(i) \\ + \frac{1}{6} \sum_{\lambda \neq \mu \neq \nu} a_{\lambda}^2 a_{\mu}^2 a_{\nu}^2 \sum_i \psi_{\lambda}^2(i) \psi_{\mu}^2(i) \psi_{\nu}^2(i).$$

We used

$$\sum_i \psi_{\lambda}^4(i) \approx 3 \sum_{i,j} \psi_{\lambda}^2(i) \psi_{\nu}^2(j) J_{ij}^2,$$

valid for $Z \gg 1$

intermode interaction terms change the coefficients of the terms quadratic and quartic in a_{λ} in the effective single-mode energy $F_{\lambda}(a_{\lambda})$. To study the influence of mode-mode coupling, we introduce the molecular field B_i ,

$$B_i = \sum_{\lambda} \langle a_{\lambda}^2 \rangle \psi_{\lambda}^2(i),$$

At $|\tau| \gg \tau_0$ fluctuations of B_i are small $B = \frac{1}{N} \sum_i B_i = \frac{1}{N} \sum_{\lambda} \langle a_{\lambda}^2 \rangle \rightarrow$

$$\frac{\partial H_0}{\partial a_{\lambda}^2} = \frac{1}{2} [(\tau + B)^2 - \epsilon_{\lambda}] + \left(\frac{1}{3} \tau + B \right) a_{\lambda}^2 \sum_i \psi_{\lambda}^4(i) = 0$$

$$a_\lambda^2 = \frac{3}{4|\tau|} (\epsilon_\lambda - \xi) V_\lambda \Theta(\epsilon_\lambda - \xi)$$

$$V_\lambda = [\sum_i \psi_\lambda^4(i)]^{-1} \text{ - localization volume}$$

here $\xi = (\tau + B)^2$

At $|\tau| \gg \tau_0$

Self-consistency eq.

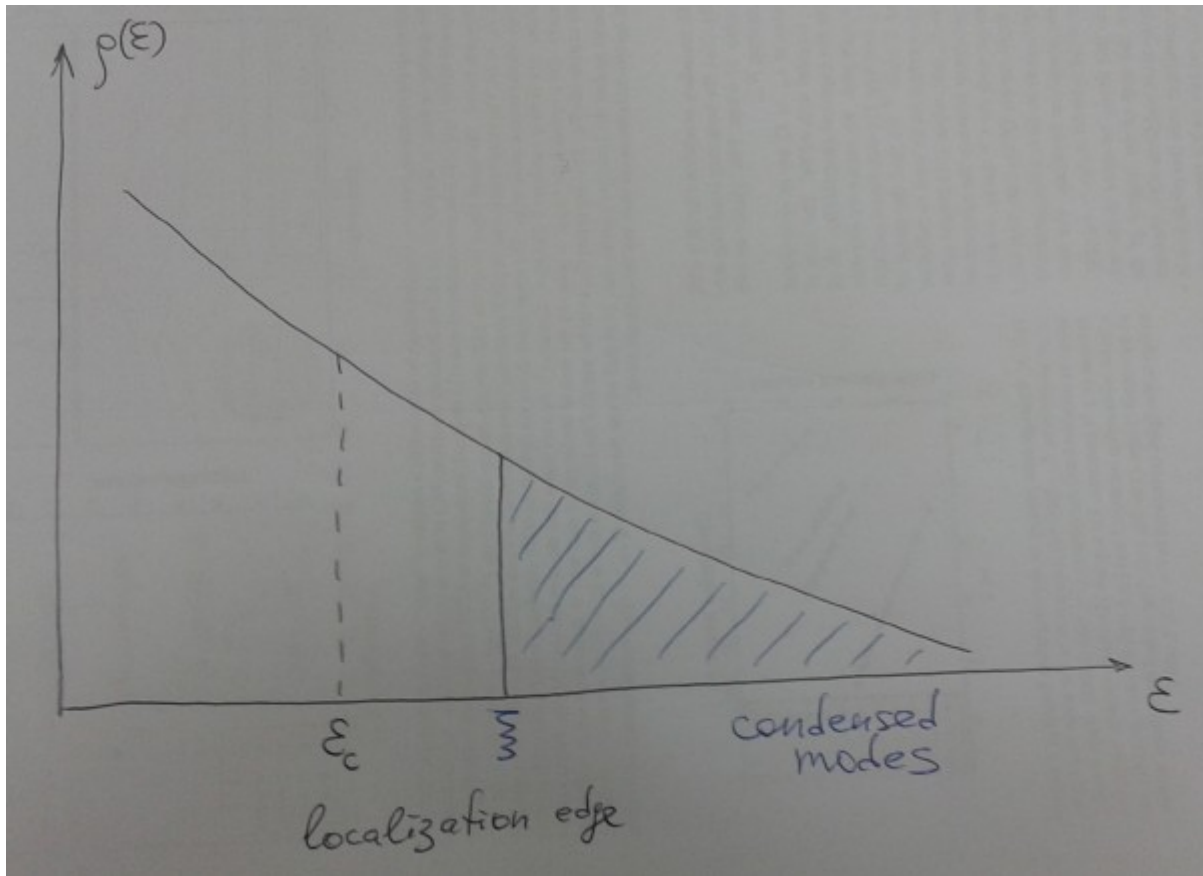
$$\tau^2 = \frac{1}{N} \frac{3}{4} \sum_\lambda (\epsilon_\lambda - \xi) V_\lambda \Theta(\epsilon_\lambda - \xi)$$

$$\approx \frac{3}{4} \int \rho(\epsilon) (\epsilon - \xi) V(\epsilon) d\epsilon$$

$$V(\epsilon) \sim Z^2 \left(\frac{\epsilon - \epsilon_c}{\epsilon_0} \right)^{-\theta}$$

For $\theta > 2$

$$\left(\frac{\tau}{\tau_0} \right)^2 \sim \zeta^{2-\theta}, \quad \zeta = \frac{\xi - \epsilon_c}{\epsilon_0}$$



ζ decreases and tends to approach zero upon T decrease

$$\theta = \nu D_2$$

Current knowledge of the exponents:

Finite-size scaling and multifractality at the Anderson transition for the three Wigner-Dyson symmetry classes in three dimensions

L up to 100

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Orthogonal ensemble:

$$D_2 = 1.231$$

$$\nu = 1.595$$



$$\theta = 1.96$$

A word of caution: we need to average $V_\lambda = [\sum_i \psi_\lambda^4(i)]^{-1}$

instead of the usual IPR $\sum_i \psi_\lambda^4(i)$

which is used to define D_2

V_λ is determined by the most extended states with small IPR

Thus we need the value of D_2 defined by typical IPR

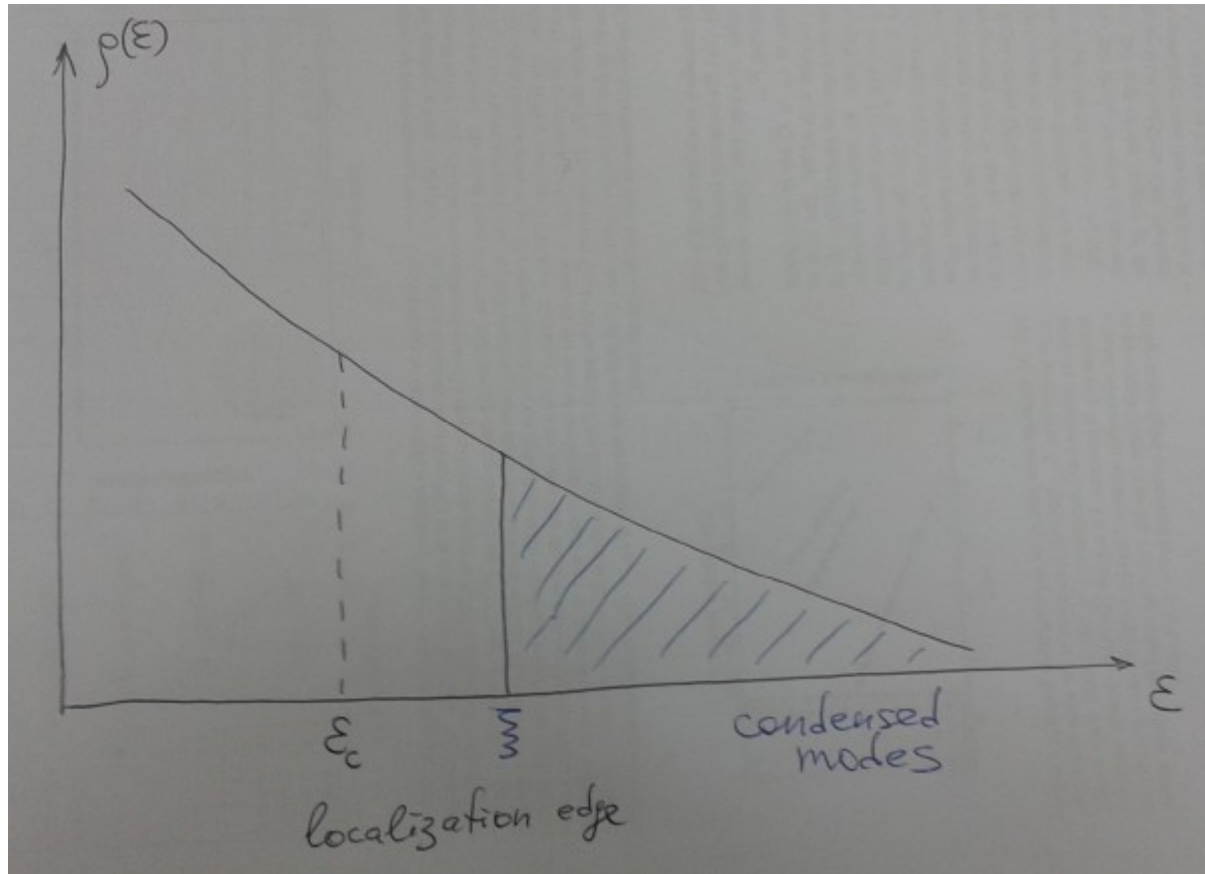
rather than average ones

It leads (Cuevas, Kravtsov 2007) to slightly large value of D_2

Assuming $\theta \approx 2$ we find a similar solution for self-consistency equation:

$$\left(\frac{\tau}{\tau_0}\right)^2 \sim \ln(1/\zeta)$$

$$\zeta = \frac{\xi - \epsilon_c}{\epsilon_0}$$



The borderline between condensed and uncondensed modes asymptotically approaches mobility edge

Effective Superspin Interaction and Hierarchy of Superparamagnets

Now we should include “odd” terms in the Hamiltonian of slow modes
 These terms are smaller than “even” ones, but they also grow
 with T and ζ decrease and eventually become relevant
 at some temperature when $T_0 - T = -\tau = \tau_1$

τ_1 is determined by the condition that effective interaction
 between signs of amplitudes a_λ i.e. $\sigma_\lambda = a_\lambda / |a_\lambda|$
 start to interact strongly

$$H_1\{\sigma_\lambda\} = -\frac{1}{2} \sum_{\lambda, \mu} I_{\lambda\mu} \sigma_\lambda \sigma_\mu.$$

At leading order in τ

$$I_{\lambda\mu} = |a_\lambda a_\mu| \left[|\tau| \sum_i \psi_\lambda(i) \psi_\mu(i) \delta B(i) + \sum_{ij} J_{ij}^2 \psi_\lambda(i) \psi_\mu(j) \delta B(i, j) \right]$$

$$\delta B_i \equiv B_i - B$$

$$\delta B(i, j) = \sum_\nu a_\nu^2 \psi_\nu(i) \psi_\nu(j)$$

Signs of $I_{\lambda\mu}$ are random i.e. this interaction is also of SG-type

Interaction of “superspins”: major parameters

$$I_{\lambda\mu} = |a_\lambda a_\mu| \left[|\tau| \sum_i \psi_\lambda(i) \psi_\mu(i) \delta B(i) + \sum_{ij} J_{ij}^2 \psi_\lambda(i) \psi_\mu(j) \delta B(i, j) \right]$$

The couplings $I_{\lambda\mu}$ are random and weakly (at most) correlated owing to the random character of the eigenfunctions $\psi_\lambda(i)$ and the “background” field δB_i .

The effective strength of the σ_λ interaction is characterized by the parameter

$$J_1^2 = \sum_\lambda I_{\lambda\mu}^2 \sim \tilde{Z} I_{\lambda\mu}^2 \quad \text{That is analog of } J_0^2 = \sum_j J_{ij}^2 \quad (\text{we put it} = 1)$$

The main contribution comes from the largest-scale modes with $\epsilon_\lambda - \xi \approx \xi - \epsilon_c = \zeta \epsilon_0$

The corresponding eigenfunctions $\psi_\lambda(i)$ overlap strongly, with mean coordination number $\tilde{Z} \gg 1$

$$\tilde{Z}(\zeta) \sim \zeta^{-\kappa}$$

$$J_1(\zeta) \sim Z^{-1} \zeta^{-\phi} + Z^{-5/6} \zeta^{-\phi'}$$

$$\delta B_i \equiv B_i - B$$

$$\delta B(i, j) = \sum_\nu a_\nu^2 \psi_\nu(i) \psi_\nu(j)$$

$$I_{\lambda\mu} = |a_\lambda a_\mu| \left[|\tau| \sum_i \psi_\lambda(i) \psi_\mu(i) \delta B(i) + \sum_{ij} J_{ij}^2 \psi_\lambda(i) \psi_\mu(j) \delta B(i, j) \right] = I_{\lambda\mu}^{(1)} + I_{\lambda\mu}^{(2)}$$

Estimate the values of $I^{(1)}$ and $I^{(2)}$ at $\tau \sim \tau_0$

$$a_\lambda^2 \sim \frac{\epsilon_\lambda - \tau_0^2}{|\tau_0|} V(\epsilon_\lambda \sim \epsilon_0) \sim Z^{4/3} \quad V(\epsilon_\lambda \sim \epsilon_0) \sim Z^2$$

$$\delta B_i \sim \tau_0 \quad \delta B(i, j) \sim \tau_0 \quad \psi_\lambda(i) \psi_\mu(j) \sim \frac{1}{V_{\lambda, \mu}}$$

$I_{\lambda\mu}^{(1)}$ contains sign-alternating sum over $\sim Z^2$ sites \longrightarrow Z factor

$I_{\lambda\mu}^{(2)}$ contains sign-alternating sums over Z^2 sites i and Z sites j (which are coupled to i)

Combining all factors we find for $\tau \sim \tau_0$ $I^{(1)} \sim Z^{-1}$ $I^{(2)} \sim Z^{-5/6}$

At lower T's both sums grow as powers of ζ :

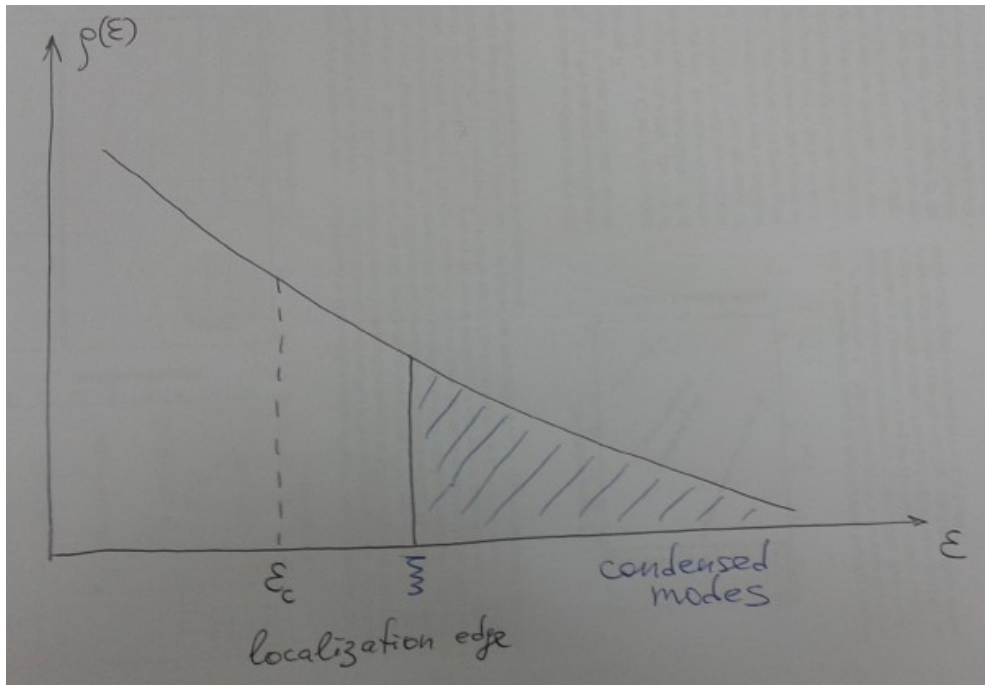
$$J_1(\zeta) \sim \max[Z^{-1} \zeta^{-\phi}, Z^{-5/6} \zeta^{-\phi'}]$$

Assume by now that the first term $I^{(1)}$ is the major one (to be checked!)

Effective coupling strength J_1 becomes of the order of unity at

$$\zeta = \zeta_1 \sim Z^{-1/\phi}$$

Effective coordination number at $\zeta = \zeta_1$ is $\tilde{Z} \sim Z^{\kappa/\phi}$



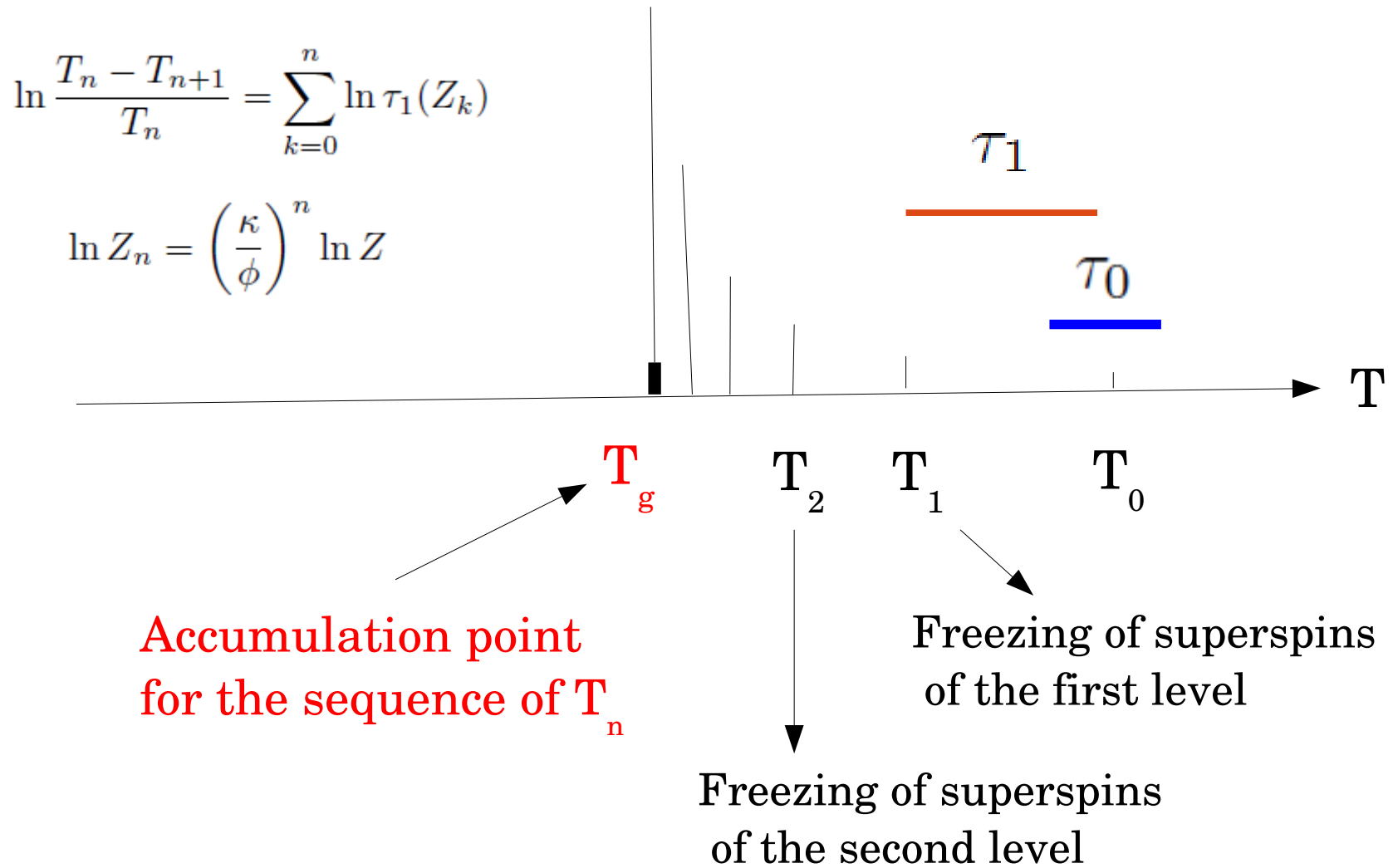
$$\zeta = \frac{\xi - \epsilon_c}{\epsilon_0}, \quad \ln \frac{1}{\zeta} \sim (\tau/\tau_0)^2$$

$$\tau_1 \sim \tau_0 \sqrt{\frac{\ln Z}{\phi}} > \tau_0$$

The key point: is the ratio κ/ϕ larger than unity ?

If $\kappa/\phi > 1$ then $\tilde{Z} > Z$ \rightarrow New SG problem is the same type as original one

Sequence of fractal clusters that grow upon temperature decrease



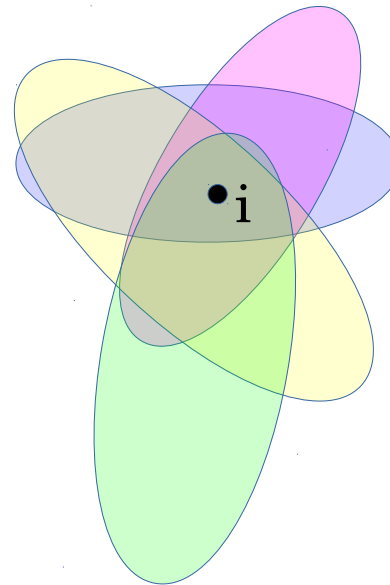
Local magnetization is a function of “super-spins” that belong to overlapping fractal clusters

$$m_i = \sum_{\lambda} a_{\lambda} \psi_{\lambda}(i)$$

$$a_{\lambda} = \sum_{\alpha} a_{\alpha}^{(2)} \psi_{\alpha}^{(2)}(\lambda)$$

$$a_{\alpha}^{(2)} = \sum_{\beta} a_{\beta}^{(3)} \psi_{\beta}^{(3)}(\alpha)$$

.....



With T decrease, the system inevitably goes out of equilibrium as barriers for largest condensed modes grow above $T \ln(t_{\text{obs}}/t_0)$

Conclusions

- Spin glass with large Z **may** present very special phase transition scenario with hierarchy of fractal clusters, instead of usually presumed scaling behavior.
- To prove (or disprove) such a scenario, a careful analysis of wave-function statistics near 3D AT is needed.
- The same approach can be used for Gauge Glasses (randomly frustrated Josephson networks)
- Generalization for the Quantum Spin Glass problem seems to be possible

Alternative scenario

If $\kappa/\phi < 1$ then $\tilde{Z} < Z$

In this case effective coordination number drops during each step of RG transformations and critical behavior is the same as in the Spin glass model with $Z \sim 1$

The values of the exponents κ and ϕ are to be determined by the statistics of critical eigenfunctions near Anderson transition