Theory of low-dimensional disordered electronic systems

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Plan (tentative)

- **Quantum interference and localization in disordered systems**
  - **diagrammatics; weak localization; mesoscopic fluctuations**
  - **field theory: non-linear σ-model**
  - **quasi-1D geometry: exact solution, localization**
  - **RG, metal-insulator transition, criticality, wave function multifractality**
- **Electron-electron interaction effects**
  - **dephasing and renormalization**
  - **interaction effects at Anderson transitions**
  - **disorder, localization, dephasing in interacting 1D (Luttinger liquid)**
- **Quantum Hall effects**
  - **IQHE: localization, plateau transitions, field theories, criticality**
  - **FQHE: overview, composite fermions**
- **Symmetries, unconventional symmetry classes**
  - **symmetry classification of disordered electronic systems**
  - **mechanisms of delocalization and criticality in quasi-1D and 2D systems**
  - **disordered Dirac fermions; electron transport in disordered graphene**

Evers, ADM, arXiv:0707.4378 "Anderson transitions", for Rev. Mod. Phys.

Basics of disorder diagrammatics

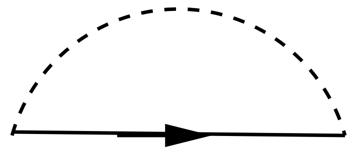
Hamiltonian $$H = H_0 + V(\mathrm{r}) \equiv \frac{(-i\nabla)^2}{2m} + V(\mathrm{r})$$

Free Green function $$G_0^{R,A}(\epsilon, p) = (\epsilon - p^2/2m \pm i0)^{-1}$$

Disorder $$\langle V(\mathrm{r})V(\mathrm{r}')\rangle = W(\mathrm{r} - \mathrm{r}')$$

simplest model: white noise $$W(\mathrm{r} - \mathrm{r}') = \Gamma\delta(\mathrm{r} - \mathrm{r}')$$

self-energy $$\Sigma(\epsilon, p)$$



$$\mathrm{Im}\,\Sigma_R = \Gamma \int (dp)\mathrm{Im}\,\frac{1}{\epsilon - p^2/2m + i0} = \pi\nu\Gamma \equiv -\frac{1}{2\tau}\ ,$$

τ – **mean free time**

disorder-averaged Green function $$G(\epsilon, p)$$

$$G^{R,A}(\epsilon, p) = \frac{1}{\epsilon - p^2/2m - \Sigma_{R,A}} \simeq \frac{1}{\epsilon - p^2/2m \pm i/2\tau}$$

$$G^{R,A}(\epsilon, r) \simeq G_0^{R,A}(\epsilon, r)e^{-r/2l}\ ,$$

$l = v_F\tau$ – **mean free path**

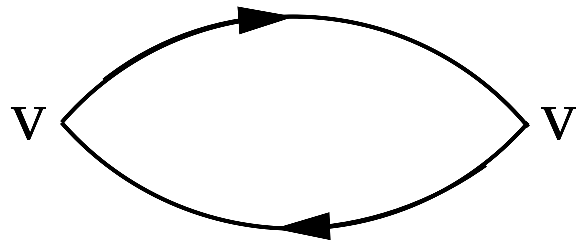
Conductivity

Kubo formula $$\sigma_{\mu\nu}(\omega) = \frac{1}{i\omega}\left\{\frac{i}{\hbar}\int_0^\infty dt \int dr e^{i\omega t}\langle[j_\mu(r,t), j_\nu(0,0)]\rangle - \frac{ne^2}{m}\delta_{\mu\nu}\right\}$$

Non-interacting electrons, $T, \omega \ll \epsilon_F$:

$$\sigma_{xx}(\omega) \simeq \frac{e^2}{2\pi V}\mathrm{Tr}\, \hat{v}_x G^R_{\epsilon+\omega}\hat{v}_x(G^A_\epsilon - G^R_\epsilon) \qquad \epsilon \equiv \epsilon_F$$

Drude conductivity:

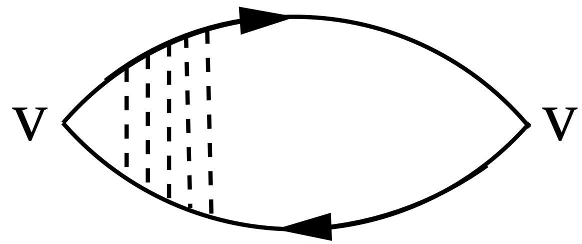


$$\sigma_{xx} = \frac{e^2}{2\pi}\int (dp)\frac{1}{m^2}p_x^2 G^R_{\epsilon+\omega}(p)[G^A_\epsilon(p) - G^R_\epsilon(p)]$$

$$\simeq \frac{e^2}{2\pi}\nu\frac{v_F^2}{d}\int d\xi_p \frac{1}{(\omega - \xi_p + \frac{i}{2\tau})(-\xi_p - \frac{i}{2\tau})} = e^2\frac{\nu v_F^2}{d}\frac{\tau}{1 - i\omega\tau}\,, \qquad \xi_p = \frac{p^2}{2m} - \epsilon$$

Finite-range disorder $\longrightarrow$ **anisotropic scattering**

$\longrightarrow$ **vertex correction** , $\quad \tau \longrightarrow \tau_{\mathrm{tr}}$

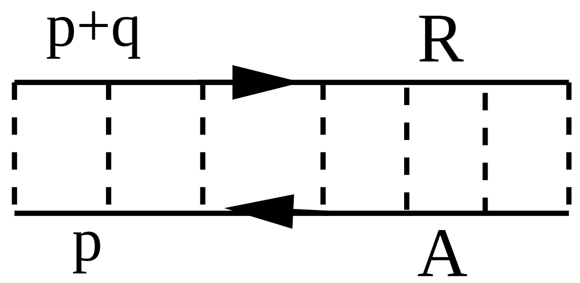


$$\frac{1}{\tau} = \nu\int\frac{d\phi}{2\pi}w(\phi) \qquad \frac{1}{\tau_{\mathrm{tr}}} = \nu\int\frac{d\phi}{2\pi}w(\phi)(1 - \cos\phi)$$

Diffuson and Cooperon

$$\mathcal{D}(q,\omega) = (2\pi\nu\tau)^{-2}\int d(r-r')\langle G^R_\epsilon(r',r)G^A_{\epsilon+\omega}(r,r')\rangle e^{-iq(r-r')}$$

Ladder diagrams (diffuson)

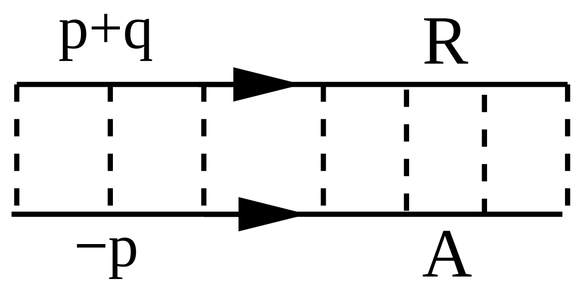


$$\frac{1}{2\pi\nu\tau}\sum_{n=0}^{\infty}\left[\frac{1}{2\pi\nu\tau}\int (dp)G^R_{\epsilon+\omega}(p+q)G^A_\epsilon(p)\right]^n$$

$$\int G^R G^A \simeq \int d\xi_p \frac{d\phi}{2\pi}\frac{1}{(\omega-\xi_p-v_F q\cos\phi+\frac{i}{2\tau})(-\xi_p-\frac{i}{2\tau})} = 2\pi\nu\tau[1-\tau(Dq^2-i\omega)]$$

$$\mathcal{D}(q,\omega) = \frac{1}{2\pi\nu\tau^2}\,\frac{1}{Dq^2-i\omega}$$ diffusion pole $\quad ql,\omega\tau \ll 1$

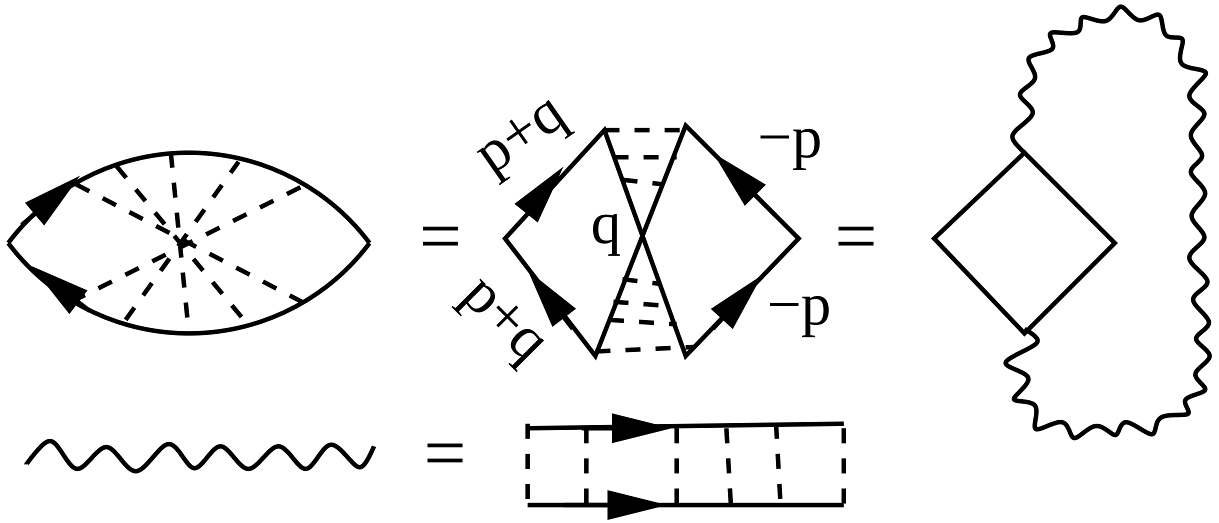
$$\left(\partial/\partial t - D\nabla_r^2\right)\mathcal{D}(r-r',t-t') = 2\pi\nu\delta(r-r')\delta(t-t')$$

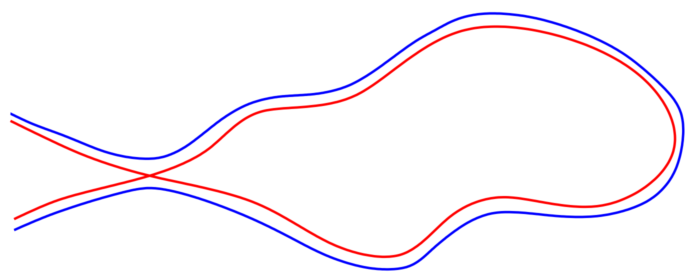


Weak. loc. correction: Cooperon $\mathcal{C}(q,\omega)$

Time-reversal symmetry preserved, no interaction $\longrightarrow$ $\mathcal{C}(q,\omega)=\mathcal{D}(q,\omega)$

Weak localization (orthogonal symmetry class)





Cooperon loop (interference of time-reversed paths)

$$\boldsymbol{\Delta\sigma_{\mathrm{WL}} \simeq -\frac{e^2}{2\pi}\frac{v_F^2}{d}\nu \int d\xi_p G_R^2 G_A^2 \int (dq)\frac{1}{2\pi\nu\tau^2}\,\frac{1}{Dq^2 - i\omega} = -\sigma_0 \frac{1}{\pi\nu}\int \frac{(dq)}{Dq^2 - i\omega}}$$

$$\boldsymbol{\Delta\sigma_{\mathrm{WL}} = -\frac{e^2}{(2\pi)^2}\left(\frac{\sim 1}{l} - \frac{1}{L_\omega}\right)}\ ,$$ 3D

$$\boldsymbol{L_\omega = \left(\frac{D}{-i\omega}\right)^{1/2}}$$

$$\boldsymbol{\Delta\sigma_{\mathrm{WL}} = -\frac{e^2}{2\pi^2}\ln\frac{L_\omega}{l}}\ ,$$ 2D

$$\boldsymbol{\Delta\sigma_{\mathrm{WL}} = -\frac{e^2}{2\pi}L_\omega}\ ,$$ quasi-1D

Generally: IR cutoff

$$\boldsymbol{L_\omega \longrightarrow \min\{L_\omega, L_\phi, L, L_H\}}$$

Strong localization

WL correction is IR-divergent in quasi-1D and 2D; becomes $\sim \sigma_0$ at a scale

$$\xi \sim 2\pi\nu D \ , \qquad \text{quasi-1D}$$

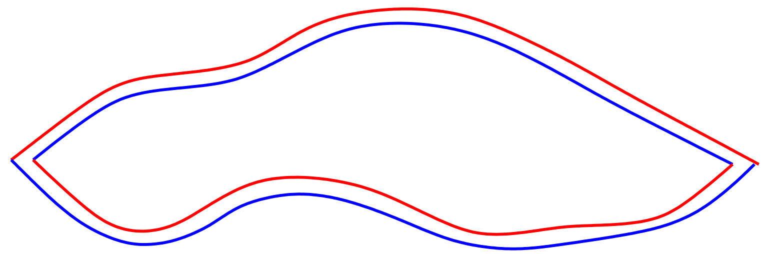
$$\xi \sim l \exp(2\pi^2\nu D) = l \exp(\pi g) \ , \qquad \text{2D}$$

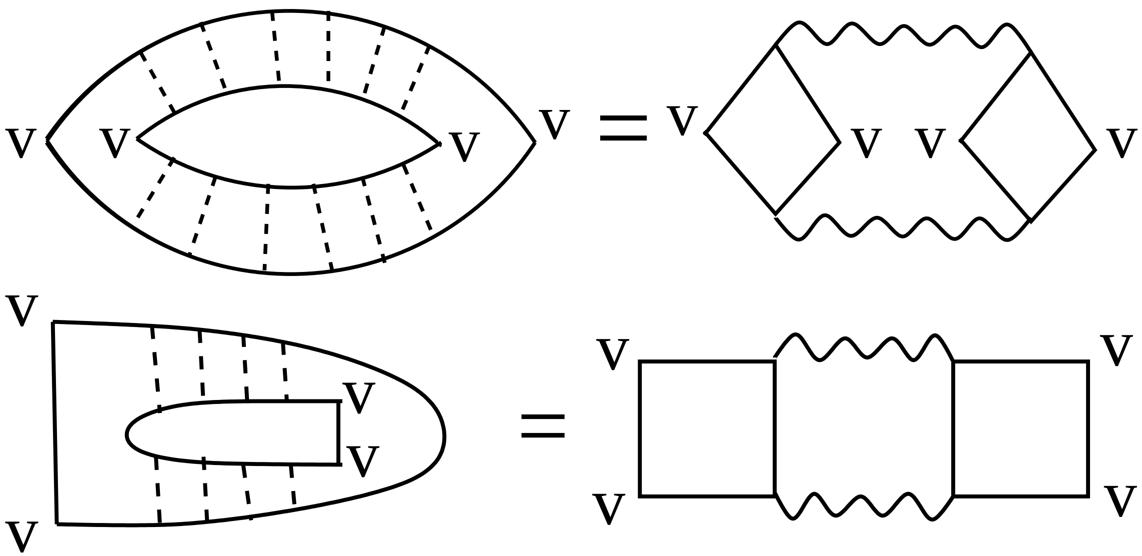
indicates strong localization, ξ – localization length

confirmed by exact solution in quasi-1D and renormalization group in 2D

Mesoscopic conductance fluctuations

$$\langle(\delta G)^2\rangle \sim \langle(\sum_{i\neq j} A_i^* A_j)^2\rangle \sim \sum_{i\neq j}\langle|A_i|^2\rangle\langle|A_j|^2\rangle$$





$$\langle(\delta\sigma)^2\rangle = 3\left(\frac{e^2}{2\pi V}\right)^2 (4\pi\nu\tau^2 D)^2 \sum_{\mathrm{q}} \left(\frac{1}{2\pi\nu\tau^2 D\mathrm{q}^2}\right)^2 = 12\left(\frac{e^2}{2\pi V}\right)^2 \sum_{\mathrm{q}}\left(\frac{1}{\mathrm{q}^2}\right)^2$$

$$\langle(\delta g)^2\rangle = \frac{12}{\pi^4}\sum_{\mathrm{n}}\left(\frac{1}{\mathrm{n}^2}\right)^2 \qquad n_x = 1, 2, 3, \ldots\ , \qquad n_{y,z} = 0, 1, 2, \ldots$$

$\langle(\delta g)^2\rangle \sim 1$ independent of system size; depends only on geometry!

$\longrightarrow$ universal conductance fluctuations (UCF)

quasi-1D geometry: $\langle(\delta g)^2\rangle = 8/15$

Mesoscopic conductance fluctuations (cont'd)

Additional comments:

- **UCF are anomalously strong from classical point of view:**

  $\langle(\delta g)^2\rangle / g^2 \sim L^{4-2d} \gg L^{-d}$

  reason: quantum coherence

- **UCF are universal for** $L \ll L_T, L_\phi$; **otherwise fluctuations suppressed**
- **symmetry dependent: 8 = 2 (Cooperons) × 4 (spin)**
- **autocorrelation function** $\langle \delta g(B) \delta g(B + \delta B) \rangle$; **magnetofingerprints**
- **mesoscopic fluctuations of various observables**

From Metal to Insulator

Philip W. Anderson

1958 “Absence of diffusion in certain random lattices”

Disorder-induced localization

$\longrightarrow$ **Anderson insulator**

$D \leq 2$ **all states are localized**

$D \geq 3$ **metal-insulator transition**

Sir Nevill F. Mott

interaction-induced gap

$\longrightarrow$ **Mott insulator**

The Nobel Prize in Physics 1977

Metal vs Anderson insulator

Localization transition $\longrightarrow$ change in behavior of diffusion propagator,

$$\Pi(\mathrm{r}_1, \mathrm{r}_2; \omega) = \langle G^R_{\epsilon+\omega/2}(\mathrm{r}_1, \mathrm{r}_2) G^A_{\epsilon-\omega/2}(\mathrm{r}_2, \mathrm{r}_1)\rangle,$$

Delocalized regime: Π has the diffusion form:

$$\Pi(\mathrm{q}, \omega) = 2\pi\nu(\epsilon)/(Dq^2 - i\omega),$$

Insulating phase: propagator ceases to have Goldstone form, becomes massive,

$$\Pi(\mathrm{r}_1, \mathrm{r}_2; \omega) \simeq \frac{2\pi\nu(\epsilon)}{-i\omega} \mathcal{F}(|\mathrm{r}_1 - \mathrm{r}_2|/\xi),$$

$\mathcal{F}(\mathrm{r})$ decays on the scale of the localization length, $\mathcal{F}(r/\xi) \sim \exp(-r/\xi)$.

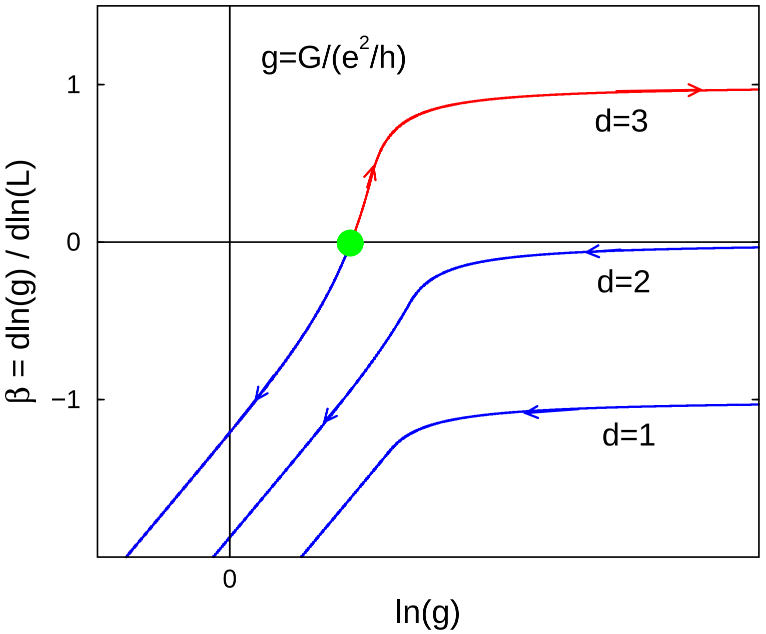
Comment:

Localization length ξ obtained from the averaged correlation function $\Pi = \langle G^R G^A \rangle$ is in general different from the one governing the exponential decay of the typical value $\Pi_{\mathrm{typ}} = \exp\langle \ln G^R G^A \rangle$.

E.g., in quasi-1D systems: $\xi_{\mathrm{av}} = 4\xi_{\mathrm{typ}}$

This is usually not important for the definition of the critical index ν.

Anderson transition

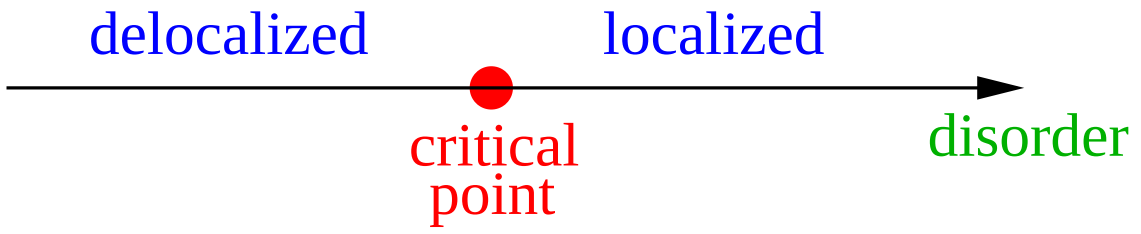


Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

Modern approach:
RG for field theory (σ-model)

quasi-1D, 2D : metallic → localized crossover with decreasing g

$d > 2$: Anderson metal-insulator transition (sometimes also in $d = 2$)



Continuous phase transition with highly unconventional properties!

Field theory: non-linear σ-model

$$S[Q] = \frac{\pi\nu}{4} \int d^d r \, \mathrm{Str}\, [-D(\nabla Q)^2 - 2i\omega\Lambda Q], \qquad Q^2(r) = 1$$

Wegner'79 (replicas); Efetov'83 (supersymmetry)

(non-equilibrium: Keldysh σ-model, will not discuss here)

σ-model manifold:

- unitary class:
  - fermionic replicas: $\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $n \to 0$
  - bosonic replicas: $\mathrm{U}(n,n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $n \to 0$
  - supersymmetry: $\mathrm{U}(1,1|2)/\mathrm{U}(1|1) \times \mathrm{U}(1|1)$
- orthogonal class:
  - fermionic replicas: $\mathrm{Sp}(4n)/\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$, $n \to 0$
  - bosonic replicas: $\mathrm{O}(2n,2n)/\mathrm{O}(2n) \times \mathrm{O}(2n)$, $n \to 0$
  - supersymmetry: $\mathrm{OSp}(2,2|4)/\mathrm{OSp}(2|2) \times \mathrm{OSp}(2|2)$

in general, in supersymmetry:

$Q \in \{$"sphere" $\times$ "hyperboloid"$\}$ "dressed" by anticommuting variables

Non-linear σ-model: Sketch of derivation

Consider unitary class for simplicity

- introduce supervector field $\Phi = (S_1, \chi_1, S_2, \chi_2)$:

$$G^R_{E+\omega/2}(r_1, r_2) G^A_{E-\omega/2}(r_2, r_1) = \int D\Phi\, D\Phi^\dagger S_1(r_1) S_1^*(r_2) S_2(r_2) S_2^*(r_1)$$
$$\times \exp\left\{ i \int dr \Phi^\dagger(r)[(E - \hat{H})\Lambda + \frac{\omega}{2} + i\eta]\Phi(r) \right\},$$

where $\Lambda = \mathrm{diag}\{1, 1, -1, -1\}$. No denominator! $Z = 1$

- disorder averaging $\longrightarrow$ quartic term $(\Phi^\dagger \Phi)^2$
- Hubbard-Stratonovich transformation:

quartic term decoupled by a Gaussian integral over a 4×4 supermatrix variable $\mathcal{R}_{\mu\nu}(r)$ conjugate to the tensor product $\Phi_\mu(r)\Phi^\dagger_\nu(r)$

- integrate out Φ fields $\longrightarrow$ action in terms of the $\mathcal{R}$ fields:

$$S[\mathcal{R}] = \pi\nu\tau \int d^d r\, \mathrm{Str}\mathcal{R}^2 + \mathrm{Str}\ln[E + (\frac{\omega}{2} + i\eta)\Lambda - \hat{H}_0 - \mathcal{R}]$$

- saddle-point approximation $\longrightarrow$ equation for $\mathcal{R}$:

$$\mathcal{R}(r) = (2\pi\nu\tau)^{-1} \langle r | (E - \hat{H}_0 - \mathcal{R})^{-1} | r \rangle$$

Non-linear σ-model: Sketch of derivation (cont'd)

The relevant set of the solutions (the saddle-point manifold) has the form:

$$\mathcal{R} = \Sigma \cdot I - (i/2\tau)Q \ , \qquad Q = T^{-1}\Lambda T \ , \qquad Q^2 = 1$$

Q – 4×4 supermatrix on the σ-model target space

- gradient expansion with a slowly varying $Q(\mathrm{r})$ $\longrightarrow$

$$\Pi(\mathrm{r}_1, \mathrm{r}_2; \omega) = \int DQ\, Q_{12}^{bb}(\mathrm{r}_1) Q_{21}^{bb}(\mathrm{r}_2) e^{-S[Q]},$$

where $S[Q]$ is the σ-model action

$$S[Q] = \frac{\pi\nu}{4} \int d^d\mathrm{r}\, \mathrm{Str}\, [-D(\nabla Q)^2 - 2i\omega\Lambda Q],$$

- size of Q-matrix: $4 = 2$ (Adv.–Ret.) $\times\ 2$ (Bose–Fermi)

- orthogonal & symplectic classes (preserved time-reversal)

$\longrightarrow$ $8 = 2$ (Adv.–Ret.) $\times\ 2$ (Bose–Fermi) $\times\ 2$ (Diff.-Coop.)

- product of N retarded and N advanced Green functions

$\longrightarrow$ σ-model defined on a larger manifold, with the base being a product of $\mathrm{U}(N,N)/\mathrm{U}(N) \times \mathrm{U}(N)$ and $\mathrm{U}(2N)/\mathrm{U}(N) \times \mathrm{U}(N)$

σ model: Perturbative treatment

For comparison, consider a ferromagnet model in an external magnetic field:

$$H[\mathrm{S}] = \int \mathrm{d}^d\mathrm{r} \left[\frac{\kappa}{2}(\nabla \mathrm{S}(\mathrm{r}))^2 - \mathrm{BS}(\mathrm{r})\right] , \qquad \mathrm{S}^2(\mathrm{r}) = 1$$

n-component vector σ-model. Target manifold: sphere $\mathrm{S}^{n-1} = \mathrm{O}(n)/\mathrm{O}(n-1)$

Independent degrees of freedom: transverse part $\mathrm{S}_\perp$; $\quad S_1 = (1 - \mathrm{S}_\perp^2)^{1/2}$

$$H[\mathrm{S}_\perp] = \frac{1}{2} \int \mathrm{d}^d\mathrm{r} \left[\kappa[\nabla \mathrm{S}_\perp(\mathrm{r})]^2 + B\mathrm{S}_\perp^2(\mathrm{r}) + O(\mathrm{S}_\perp^4(\mathrm{r}))\right]$$

Ferromagnetic phase: symmetry is broken; Goldstone modes – spin waves:

$$\langle \mathrm{S}_\perp \mathrm{S}_\perp \rangle_q \propto \frac{1}{\kappa \mathrm{q}^2 + B}$$

$$Q = \left(1 - \frac{W}{2}\right) \Lambda \left(1 - \frac{W}{2}\right)^{-1} = \Lambda \left(1 + W + \frac{W^2}{2} + \ldots\right) ; \qquad W = \begin{pmatrix} 0 & W_{12} \\ W_{21} & 0 \end{pmatrix}$$

$$S[W] = \frac{\pi\nu}{4} \int \mathrm{d}^d\mathrm{r}\, \mathrm{Str}[D(\nabla W)^2 - i\omega W^2 + O(W^3)]$$

theory of “interacting” diffusion modes. Goldstone mode: diffusion propagator

$$\langle W_{12} W_{21} \rangle_{q,\omega} \sim \frac{1}{\pi\nu(D\mathrm{q}^2 - i\omega)}$$

Quasi-1D geometry: Exact solution of the σ-model

quasi-1D geometry (many-channel wire) $\longrightarrow$ 1D σ-model

$\longrightarrow$ "quantum mechanics" , longitudinal coordinate – (imaginary) "time"

$\longrightarrow$ "Schroedinger equation" of the type $\partial_t W = \Delta_Q W$, $t = x/\xi$

- Localization length, diffusion propagator Efetov, Larkin '83
- Exact solution for the statistics of eigenfunctions Fyodorov, ADM '92-94
- Exact $\langle g\rangle(L/\xi)$ and $\mathrm{var}(g)(L/\xi)$ Zirnbauer, ADM, Müller-Groeling '92-94

e.g. for orthogonal symmetry class:

$$\langle g^n\rangle(L) = \frac{\pi}{2}\int_0^\infty d\lambda \tanh^2(\pi\lambda/2)(\lambda^2+1)^{-1} p_n(1,\lambda,\lambda)\exp\left[-\frac{L}{2\xi}(1+\lambda^2)\right]$$
$$+\ 2^4 \sum_{l\in 2N+1}\int_0^\infty d\lambda_1 d\lambda_2\, l(l^2-1)\lambda_1\tanh(\pi\lambda_1/2)\lambda_2\tanh(\pi\lambda_2/2)$$
$$\times\, p_n(l,\lambda_1,\lambda_2)\prod_{\sigma,\sigma_1,\sigma_2=\pm1}(-1+\sigma l+i\sigma_1\lambda_1+i\sigma_2\lambda_2)^{-1}\exp\left[-\frac{L}{4\xi}(l^2+\lambda_1^2+\lambda_2^2+1)\right]$$

$$p_1(l,\lambda_1,\lambda_2) = l^2+\lambda_1^2+\lambda_2^2+1,$$
$$p_2(l,\lambda_1,\lambda_2) = \frac{1}{2}(\lambda_1^4+\lambda_2^4+2l^4+3l^2(\lambda_1^2+\lambda_2^2)+2l^2-\lambda_1^2-\lambda_2^2-2)$$

Quasi-1D geometry: Exact solution of the σ-model (cont'd)

$L \ll \xi$ asymptotics:

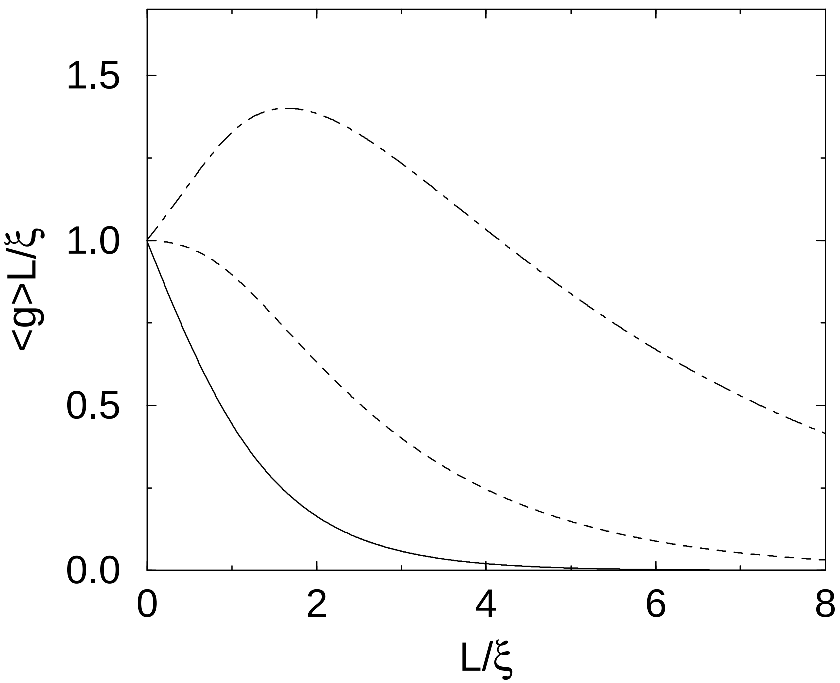
$$\langle g\rangle(L) = \frac{2\xi}{L} - \frac{2}{3} + \frac{2}{45}\frac{L}{\xi} + \frac{4}{945}\left(\frac{L}{\xi}\right)^2 + O\left(\frac{L}{\xi}\right)^3$$

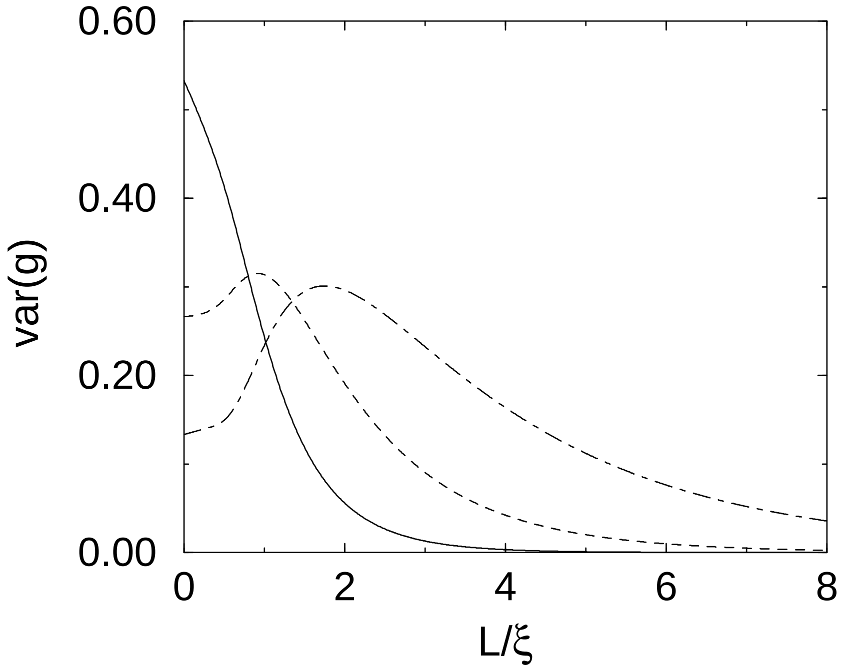
and

$$\mathrm{var}(g(L)) = \frac{8}{15} - \frac{32}{315}\frac{L}{\xi} + O\left(\frac{L}{\xi}\right)^2 .$$

$L \gg \xi$ asymptotics:

$$\langle g^n\rangle = 2^{-3/2-n}\pi^{7/2}(\xi/L)^{3/2}e^{-L/2\xi}$$





orthogonal (full), unitary (dashed), symplectic (dot-dashed)

Renormalization group and ϵ–expansion

analytical treatment of Anderson transition:

RG and ϵ-expansion for $d = 2 + \epsilon$ dimensions

β-function $\quad \beta(t) = -\dfrac{dt}{d \ln L}; \quad t = 1/2\pi g$, $\qquad g$ – dimensionless conductance

orthogonal class (preserved spin and time reversal symmetry):

$$\beta(t) = \epsilon t - 2t^2 - 12\zeta(3)t^5 + O(t^6) \qquad \text{beta-function}$$

$$t_* = \frac{\epsilon}{2} - \frac{3}{8}\zeta(3)\epsilon^4 + O(\epsilon^5) \qquad \text{transition point}$$

$$\nu = -1/\beta'(t_*) = \epsilon^{-1} - \frac{9}{4}\zeta(3)\epsilon^2 + O(\epsilon^3) \qquad \text{localization length exponent}$$

$$s = \nu\epsilon = 1 - \frac{9}{4}\zeta(3)\epsilon^3 + O(\epsilon^4) \qquad \text{conductivity exponent}$$

Numerics for 3D: $\quad \nu \simeq 1.57 \pm 0.02 \qquad$ Slevin, Othsuki ’99

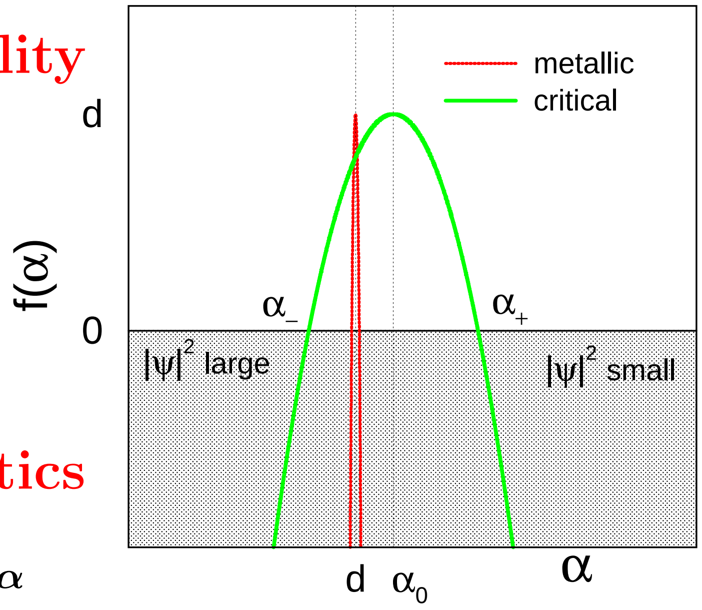
Multifractality at the Anderson transition

$P_q = \int d^d r |\psi(\mathrm{r})|^{2q}$ inverse participation ratio

$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{insulator} \\ L^{-\tau_q} & \text{critical} \\ L^{-d(q-1)} & \text{metal} \end{cases}$$

$\tau_q = \quad d(q-1) \quad + \quad \Delta_q \quad \equiv \quad D_q(q-1)$ multifractality

normal anomalous



$\tau_q \longrightarrow$ Legendre transformation

$\longrightarrow$ singularity spectrum $f(\alpha)$

$\mathcal{P}(\ln|\psi^2|) \sim L^{-d+f(\ln|\psi^2|/\ln L)}$ wave function statistics

$L^{f(\alpha)}$ – measure of the set of points where $|\psi|^2 \sim L^{-\alpha}$

Multifractality is characteristic for a variety of complex systems:

turbulence, strange attractors, diffusion-limited aggregation, . . .

Statistical ensemble $\longrightarrow$ $f(\alpha)$ may become negative

Multifractality and the field theory

Δ_q – scaling dimensions of operators $\mathcal{O}^{(q)} \sim (Q\Lambda)^q$

$d = 2 + \epsilon$: $\Delta_q = -q(q-1)\epsilon + O(\epsilon^4)$ Wegner '80

- Infinitely many operators with negative scaling dimensions
- $\Delta_1 = 0 \longleftrightarrow \langle Q \rangle = \Lambda$ naive order parameter uncritical

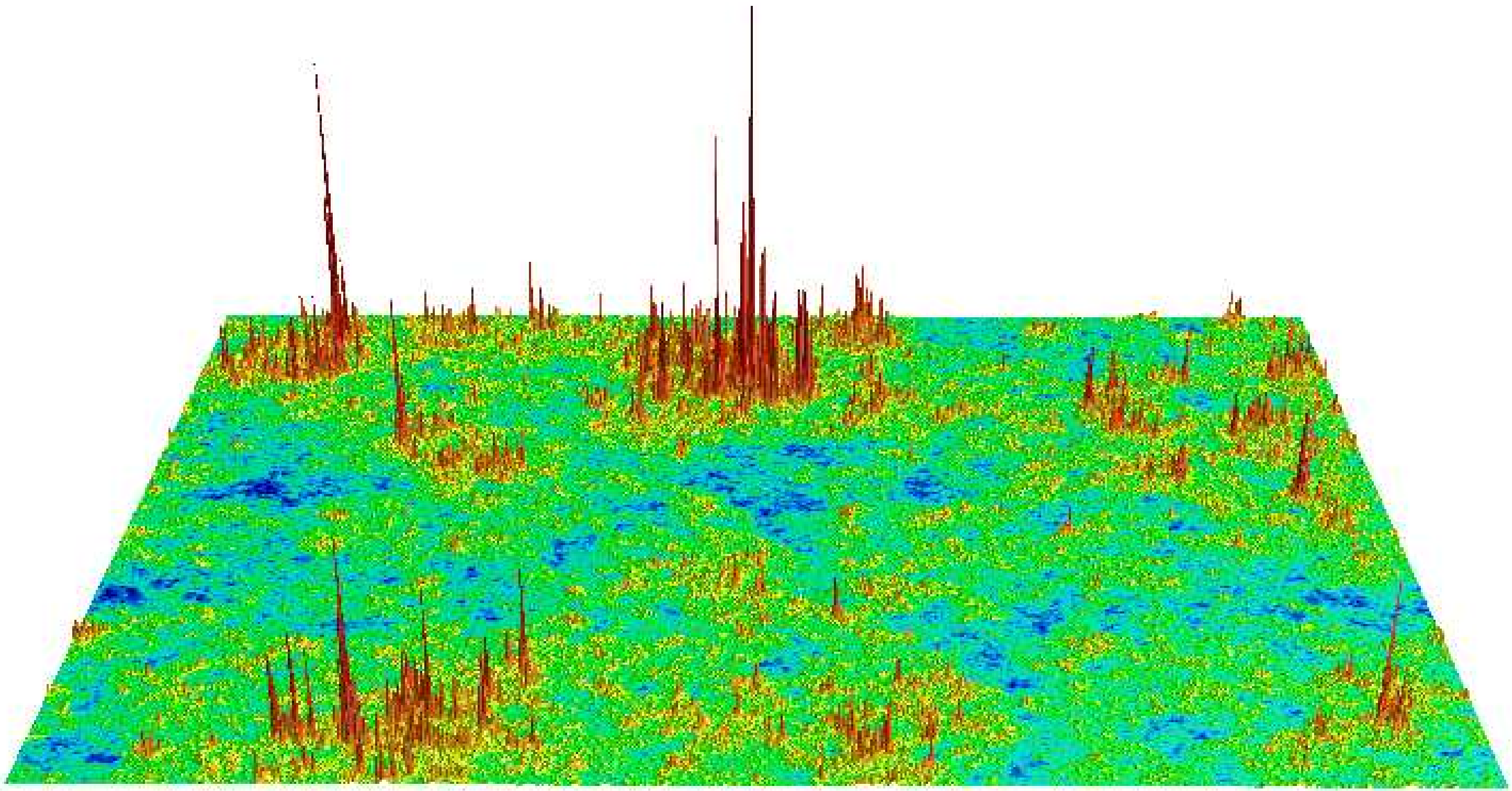
Transition described by an order parameter function $F(Q)$

Zirnbauer 86, Efetov 87

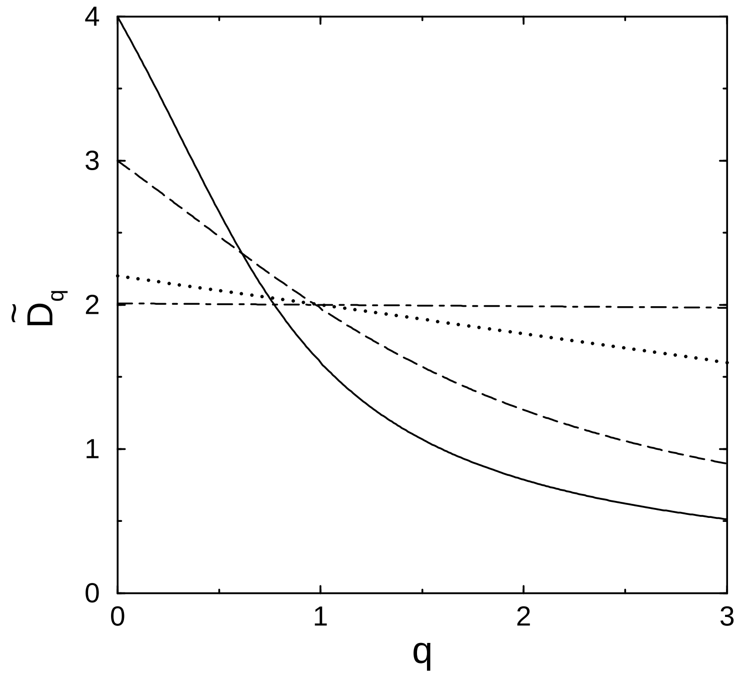
$\longleftrightarrow$ distribution of local Green functions and wave function amplitudes

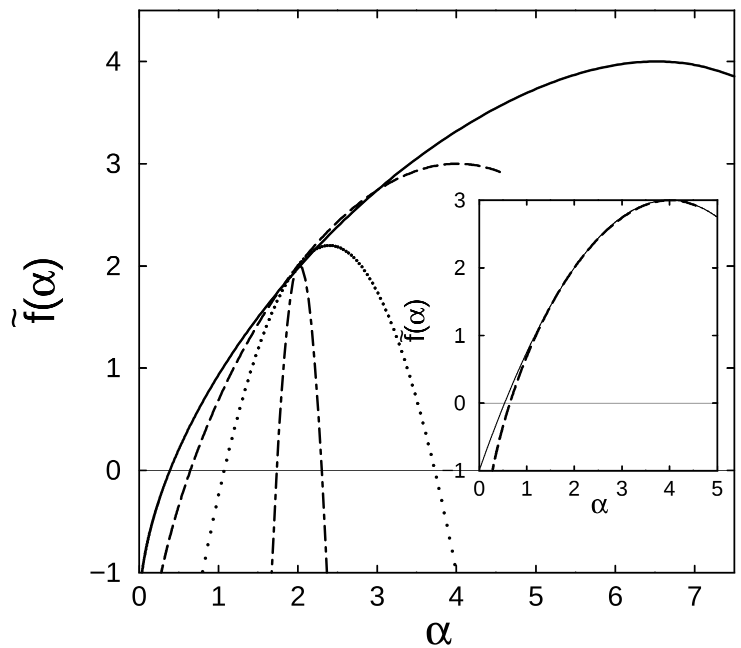
ADM, Fyodorov '91

Multifractal wave functions at the Quantum Hall transition



Dimensionality dependence of multifractality





Analytics ($2+\epsilon$, one-loop) and numerics

$$\tau_q = (q-1)d - q(q-1)\epsilon + O(\epsilon^4)$$

$$f(\alpha) = d - (d+\epsilon-\alpha)^2/4\epsilon + O(\epsilon^4)$$

$d = 4$ (full)
$d = 3$ (dashed)
$d = 2+\epsilon, \quad \epsilon = 0.2$ (dotted)
$d = 2+\epsilon, \quad \epsilon = 0.01$ (dot-dashed)

Inset: $d = 3$ (dashed)
vs. $d = 2+\epsilon, \quad \epsilon = 1$ (full)

Mildenberger, Evers, ADM '02

Power-law random banded matrix model (PRBM)

Anderson transition: dimensionality dependence:

$d = 2 + \epsilon$: weak disorder/coupling $\qquad$ $d \gg 1$: strong disorder/coupling

Evolution from weak to strong coupling – ?

PRBM $\qquad$ ADM, Fyodorov, Dittes, Quezada, Seligman '96

$N \times N$ random matrix $H = H^\dagger$ $\qquad$ $\langle |H_{ij}|^2 \rangle = \dfrac{1}{1 + |i-j|^2/b^2}$

$\longleftrightarrow$ 1D model with $1/r$ long range hopping $\qquad$ $0 < b < \infty$ parameter

Critical for any b $\longrightarrow$ family of critical theories!

$b \gg 1$ analogous to $d = 2 + \epsilon$ $\qquad$ $b \ll 1$ analogous to $d \gg 1$ (?)

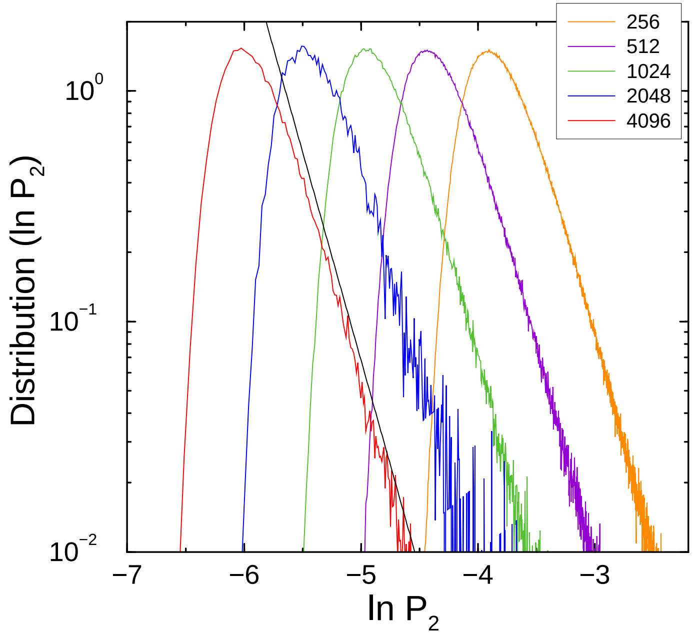
Analytics: $\quad$ $b \gg 1$: σ-model RG

$\qquad$ $b \ll 1$: real space RG

Numerics: $\quad$ efficient in a broad range of b

Evers, ADM '01

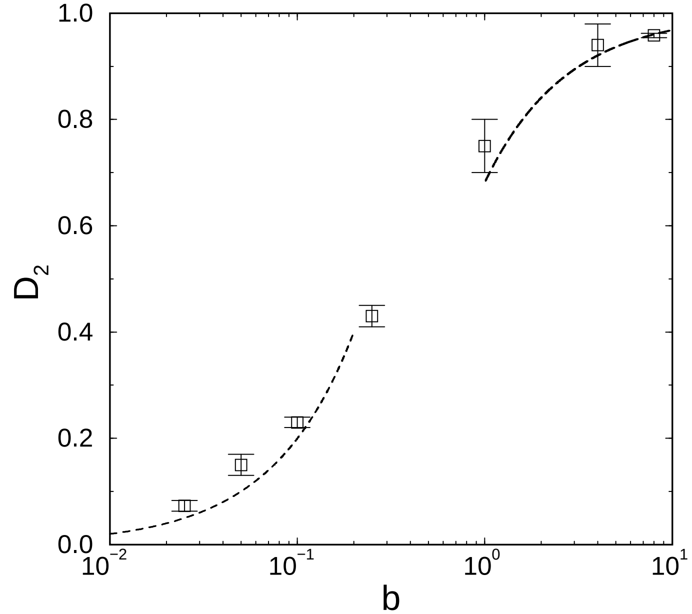
Multifractality in Power-law random banded matrix model

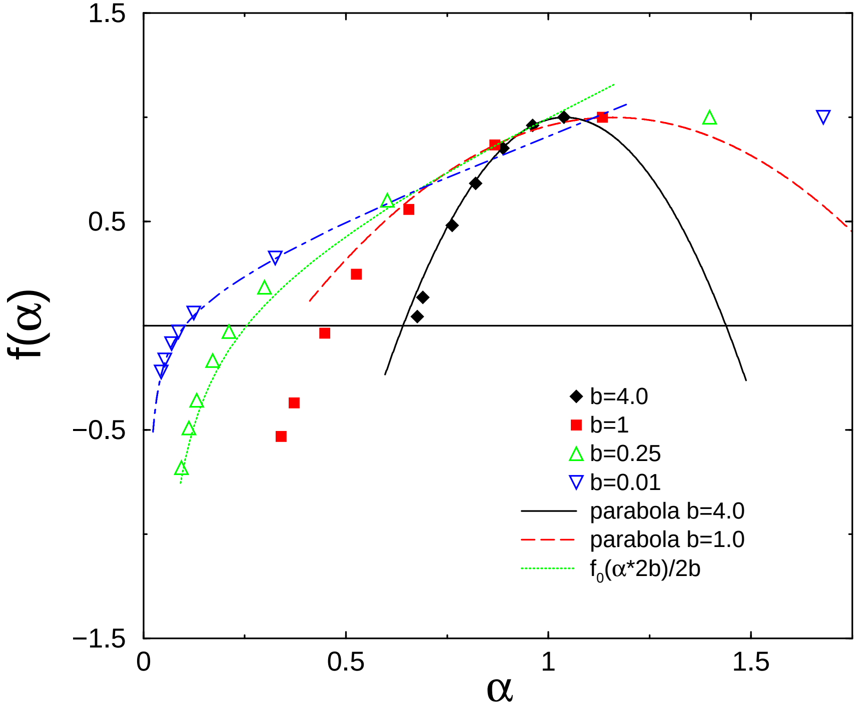


evolution of $\mathcal{P}(\ln P_2)$ with N for $b = 1$

- scale invariance
- fractal dimension $D_2 \simeq 0.75$

fractal dimension $D_2(b)$ $\longrightarrow$





Multifractality spectrum $f(\alpha)$

for $b = 4.0$, 1.0, 0.25 and 0.01

Lines: analytics for $b \gg 1$ and $b \ll 1$

Symbols: numerics

Evers, ADM '01