Low-dimensional disordered electronic systems

(theoretical aspects)

Part III

Alexander D. Mirlin

Research Center Karslruhe & University Karlsruhe & PNPI St. Petersburg

http://www-tkm.physik.uni-karlsruhe.de/∼mirlin/

Evers, ADM “Anderson transitions”, arXiv:0707.4378

Disordered electronic systems: Symmetry classification

Altland, Zirnbauer '97

Conventional (Wigner-Dyson) classes

	T	spin rot.	chiral	p-h	symbol
GOE	$+$	$+$	$-$	$-$	AI
GUE	$-$	$+/-$	$-$	$-$	A
GSE	$+$	$-$	$-$	$-$	AII

Chiral classes

	T	spin rot.	chiral	p-h	symbol
ChOE	$+$	$+$	$+$	$-$	BDI
ChUE	$-$	$+/-$	$+$	$-$	AIII
ChSE	$+$	$-$	$+$	$-$	CII

$$\boldsymbol{H} = \begin{pmatrix} 0 & t \\ t^\dagger & 0 \end{pmatrix}$$

Bogoliubov-de Gennes classes

	T	spin rot.	chiral	p-h	symbol
	$+$	$+$	$-$	$+$	CI
	$-$	$+$	$-$	$+$	C
	$+$	$-$	$-$	$+$	DIII
	$-$	$-$	$-$	$+$	D

$$\boldsymbol{H} = \begin{pmatrix} h & \boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^* & -h^T \end{pmatrix}$$

Disordered electronic systems: Symmetry classification

Ham. class	RMT	T	S	compact symmetric space	non-compact symmetric space	σ-model B|F	σ-model compact sector $\mathcal{M}_F$
Wigner-Dyson classes							
A	GUE	$-$	$\pm$	$\mathrm{U}(N)$	$\mathrm{GL}(N,\mathbb{C})/\mathrm{U}(N)$	AIII|AIII	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$
AI	GOE	$+$	$+$	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathrm{GL}(N,\mathbb{R})/\mathrm{O}(N)$	BDI|CII	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$
AII	GSE	$+$	$-$	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathrm{U}^*(2N)/\mathrm{Sp}(2N)$	CII|BDI	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$
chiral classes							
AIII	chGUE	$-$	$\pm$	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$	$\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$	A|A	$\mathrm{U}(n)$
BDI	chGOE	$+$	$+$	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\mathrm{SO}(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	AI|AII	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$
CII	chGSE	$+$	$-$	$\mathrm{Sp}(2p+2q)/\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$	$\mathrm{Sp}(2p,2q)/\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$	AII|AI	$\mathrm{U}(n)/\mathrm{O}(n)$
Bogoliubov - de Gennes classes							
C		$-$	$+$	$\mathrm{Sp}(2N)$	$\mathrm{Sp}(2N,\mathbb{C})/\mathrm{Sp}(2N)$	DIII|CI	$\mathrm{Sp}(2n)/\mathrm{U}(n)$
CI		$+$	$+$	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	$\mathrm{Sp}(2N,\mathbb{R})/\mathrm{U}(N)$	D|C	$\mathrm{Sp}(2n)$
BD		$-$	$-$	$\mathrm{SO}(N)$	$\mathrm{SO}(N,\mathbb{C})/\mathrm{SO}(N)$	CI|DIII	$\mathrm{O}(2n)/\mathrm{U}(n)$
DIII		$+$	$-$	$\mathrm{SO}(2N)/\mathrm{U}(N)$	$\mathrm{SO}^*(2N)/\mathrm{U}(N)$	C|D	$\mathrm{O}(n)$

Disordered wires: DMPK approach

Dorokhov '82 ; Mello, Pereyra, Kumar '88

transfer matrix $\begin{pmatrix} R^{\rm out} \\ R^{\rm in} \end{pmatrix} = M \begin{pmatrix} L^{\rm in} \\ L^{\rm out} \end{pmatrix}$

current conservation:

$|R^{\rm out}|^2 - |R^{\rm in}|^2 = |L^{\rm in}|^2 - |L^{\rm out}|^2 \quad \longrightarrow \quad M \in G = {\rm U}(N,N)$

Cartan decomposition $M = \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix} \begin{pmatrix} \cosh\hat{x} & \sinh\hat{x} \\ \sinh\hat{x} & \cosh\hat{x} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}$

$\hat{x} = {\rm diag}(x_1, \ldots, x_N)$ — "radial coordinates";

left/right matrices ("angular coordinates") — elements of $K = {\rm U}(N)\times{\rm U}(N)$

$\longrightarrow$ Brownian motion on G/K , described by DMPK equation

$$\frac{d\mathcal{P}}{dL} = \frac{1}{2\ell\gamma} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} J(x) \frac{\partial}{\partial x_i} J^{-1}(x)\ \mathcal{P}\ .$$

$J(x)$ – Jacobian of transformation to the radial coordinates

Disordered wires: DMPK approach (cont'd)

WD and BdG classes: $\quad 0 < x_1 < x_2 < \ldots < x_N,$

$$J(x) = \prod_{i<j}^{N} \prod_{\pm} |\sinh(x_i \pm x_j)|^{m_o} \prod_{k}^{N} |\sinh 2x_k|^{m_l} \prod_{l}^{N} |\sinh x_l|^{m_s}$$

Chiral classes: $\quad x_1 < x_2 < \ldots < x_N,$

$$J(x) = \prod_{i<j}^{N} |\sinh(x_i - x_j)|^{m_o}$$

$$\gamma = \begin{cases} m_o(N-1) + m_l + 1 \ , & \text{WD and BdG} \ ; \\ \frac{1}{2}[2 + m_o(N-1)] \ , & \text{chiral} \ . \end{cases}$$

m_0, m_l, m_s — multiplicities of ordinary, long, and short roots.

Conductance $\quad G = s \sum_{n=1}^{N} T_n \ , \qquad T_n = \dfrac{1}{\cosh^2 x_n} \ , \qquad s$ – degeneracy

Disordered wires: Transfer matrix spaces

Ham. class	transfer matrix symmetric space		tr.matr. class	m_o	m_l	m_s
Wigner-Dyson classes						
A	$U(p+q)/U(p)\times U(q)$		AIII	2	1	$2\|p-q\|$
AI	$Sp(2N,\mathbb{R})/U(N)$		CI	1	1	0
AII	$SO^*(2N)/U(N)$	N even N odd	DIII-e DIII-o	4	1	0 4
chiral classes						
AIII	$GL(N,\mathbb{C})/U(N)$		A	2	0	0
BDI	$GL(N,\mathbb{R})/O(N)$		AI	1	0	0
CII	$U^*(2N)/Sp(2N)$		AII	4	0	0
Bogoliubov - de Gennes classes						
C	$Sp(2p,2q)/Sp(2p)\times Sp(2q)$		CII	4	3	$4\|p-q\|$
CI	$Sp(2N,\mathbb{C})/Sp(2N)$		C	2	2	0
BD	$O(p,q)/O(p)\times O(q)$		BDI	1	0	$\|p-q\|$
DIII	$SO(N,\mathbb{C})/SO(N)$	N even N odd	D B	2	0	0 2

Disordered wires: Conventional localization

standard localization, Wigner-Dyson classes $(m_l = 1,\ m_o = \beta,\ m_s = 0)$.

long-wire limit: $\quad 1 \gg T_1 \gg T_2 \gg \ldots$

$$\frac{d\mathcal{P}(x_k)}{dL} = \frac{1}{2\gamma\ell}\frac{\partial^2\mathcal{P}}{\partial x_k^2} - \frac{1}{\xi_k}\frac{\partial\mathcal{P}}{\partial x_k}, \qquad \xi_k^{-1} = [1 + \beta(k-1)]/\gamma\ell$$

advection-diffusion equation, diffusion constant $1/2\gamma\ell$, drift velocity $1/\xi_1$

solution: Gaussian form with $\langle x_k \rangle = L/\xi_k$, $\mathrm{var}(x_k) = L/\gamma\ell$

$\longrightarrow$ log of conductance $g \simeq s/\cosh^2 x_1$ has Gaussian distribution with

$$-\langle \ln g \rangle = 2L/\gamma\ell \ ; \qquad \mathrm{var}(\ln g) = 4L/\gamma\ell \ .$$

On the side of atypically large g the distribution is cut at $g \sim 1$;

average conductance is determined by this cutoff: $-\ln\langle g \rangle = L/2\gamma l$.

Typical and average localization length: $\quad \xi_{\mathrm{typ}} = \gamma\ell \ ; \qquad \xi_{\mathrm{av}} = 4\gamma\ell$.

BdG classes C, CI – similar behavior

Delocalization in disordered wires: Chiral classes

Chiral classes (AIII, BDI, CII), **BdG classes** (BD, DIII):

$m_l = 0 \quad \longrightarrow \quad$ no repulsion between x_i and $-x_i$

odd number of channels $N \quad \longrightarrow \quad \langle x_{(N+1)/2} \rangle = 0 \quad \longrightarrow$

$$-\langle \ln g \rangle = \left(\frac{8L}{\pi \gamma \ell} \right)^{1/2} ; \quad \mathrm{var}(\ln g) = \left(4 - \frac{8}{\pi} \right) \frac{L}{\gamma \ell} ;$$

$$\langle g \rangle = (2\gamma\ell/\pi L)^{1/2} ; \qquad \mathrm{var}(g) = (8\gamma\ell/9\pi L)^{1/2}.$$

- stretched-exponential decay of typical conductance
- very strong fluctuations
- very slow $L^{-1/2}$ decay of average conductance (slower than Ohm's law!)

Slightly away from criticality ($E \neq 0$):

Localization length $\qquad \xi_{\mathrm{typ}} \sim |\ln E| ; \qquad \xi_{\mathrm{av}} \sim |\ln E|^2$

DOS $\qquad \rho(E) \sim 1/|E \ln^3 E|$

Dyson '53; ...

Disordered wires with perfectly conducting channels

$m_s \neq 0 \longrightarrow$ repulsion from zero $\longrightarrow$ zero eigenvalue(s) $x_i = 0$

$\longrightarrow$ perfectly transmitting channels

Two types:

- classes A, C, BD:

arbitrary number ($|p - q|$) of perfectly transmitting modes.

Realization: Quantum Hall edge states (IQHE, SQHE, TQHE)

- classes AII, DIII : single perfectly transmitting mode

σ-model language: $\pi_1(\mathcal{M}_F) = \mathbb{Z}_2 \longrightarrow$ topological term with $\theta = \pi$ allowed.

Realizations: models with spin-orbit interaction and odd number of channels:

- nanotubes (quasi-1D graphene-based structures) with decoupled valleys
- edge states in quantum spin Hall effect

Zirnbauer'92; ADM, Müller-Groeling, Zirnbauer'94; Ando, Suzuura '02;

Takane'04; Kane and Mele '05; Bernevig, Hughes, Zhang, Science'06

Mechanisms of Anderson criticality in 2D

"Common wisdom": all states are localized in 2D

In fact, in 9 out of 10 symmetry classes the system can escape localization!

$\longrightarrow$ variety of critical points

Mechanisms of delocalization & criticality in 2D:

- broken spin-rotation invariance $\longrightarrow$ antilocalization, metallic phase, MIT
classes AII, D, DIII

- topological term $\pi_2(\mathcal{M}) = \mathbb{Z}$ (quantum-Hall-type)
classes A, C, D : IQHE, SQHE, TQHE

- topological term $\pi_2(\mathcal{M}) = \mathbb{Z}_2$
classes AII, CII

- chiral classes: vanishing β-function, line of fixed points
classes AIII, BDI, CII

- Wess-Zumino term (random Dirac fermions, related to chiral anomaly)
classes AIII, CI, DIII

Spin quantum Hall effect

- disordered d-wave superconductor (class C):

charge not conserved but spin conserved

- time-reversal invariance broken:
  - $d_{x^2-y^2} + id_{xy}$ order parameter
  - strong magnetic field
  - Haldane-Rezayi d-wave paired state of composite fermions at $\nu = 1/2$

$\longrightarrow$ SQH plateau transition: spin Hall conductivity quantized

$$j_x^Z = \sigma_{xy}^s \left(-\frac{dB^z(y)}{dy} \right)$$

Model: $SU(2)$ modification of the Chalker-Coddington network

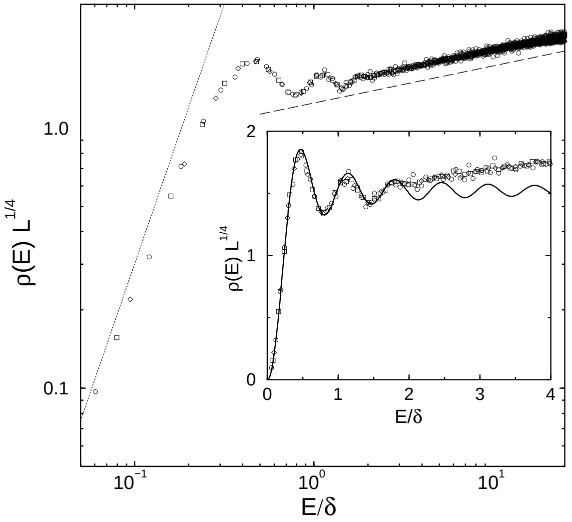
Kagalovsky, Horovitz, Avishai, Chalker '99 ; Senthil, Marston, Fisher '99

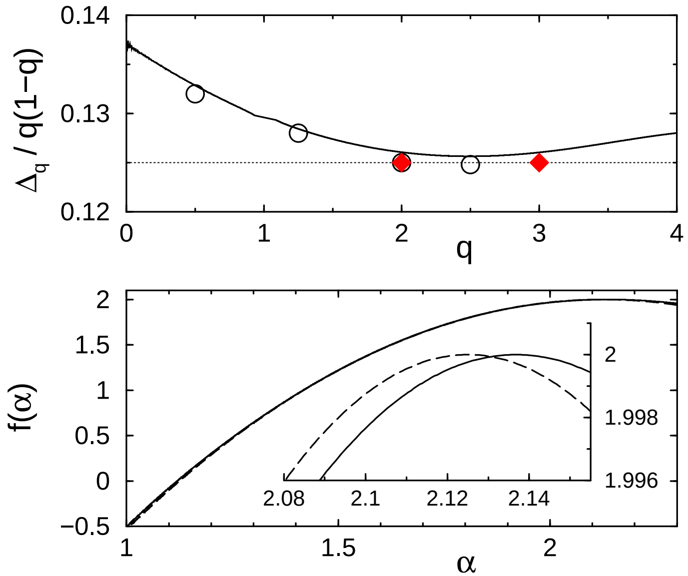
Spin quantum Hall effect (cont'd)

Similar to IQH transition but:

- DoS critical $\rho(E) \propto E^\mu$
- mapping to percolation: analytical evaluation of
  - DOS exponent $\mu = 1/7$
  - localization length exponent $\nu = 4/3$
  - lowest multifractal exponents: $\Delta_2 = -1/4, \ \Delta_3 = -3/4$
- numerics: analytics confirmed

  multifractality spectrum: Δ_q, $f(\alpha)$ not parabolic

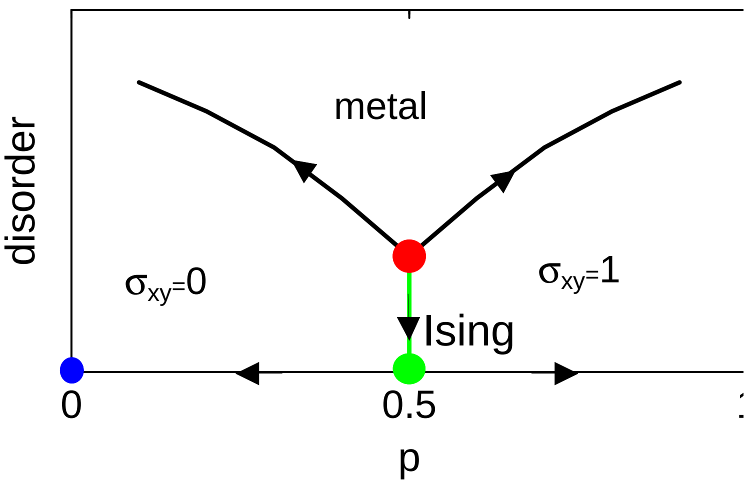


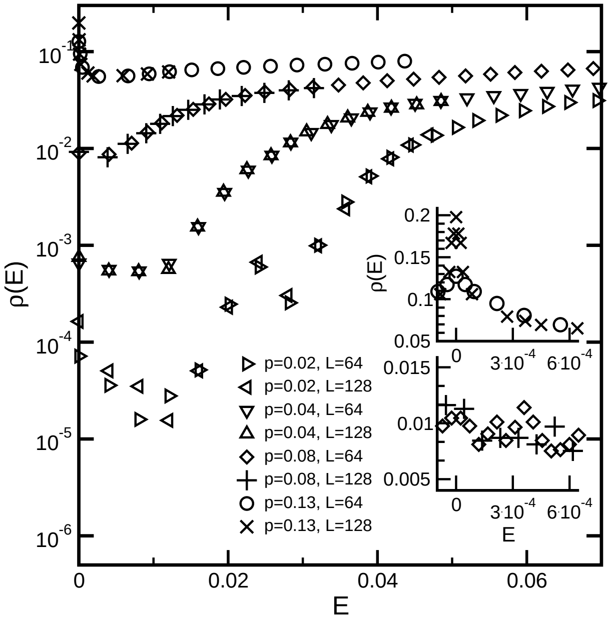


Gruzberg, Ludwig, Read '99; Beamond, Cardy, Chalker '02; Evers, Mildenberger, ADM '03

Thermal quantum Hall effect

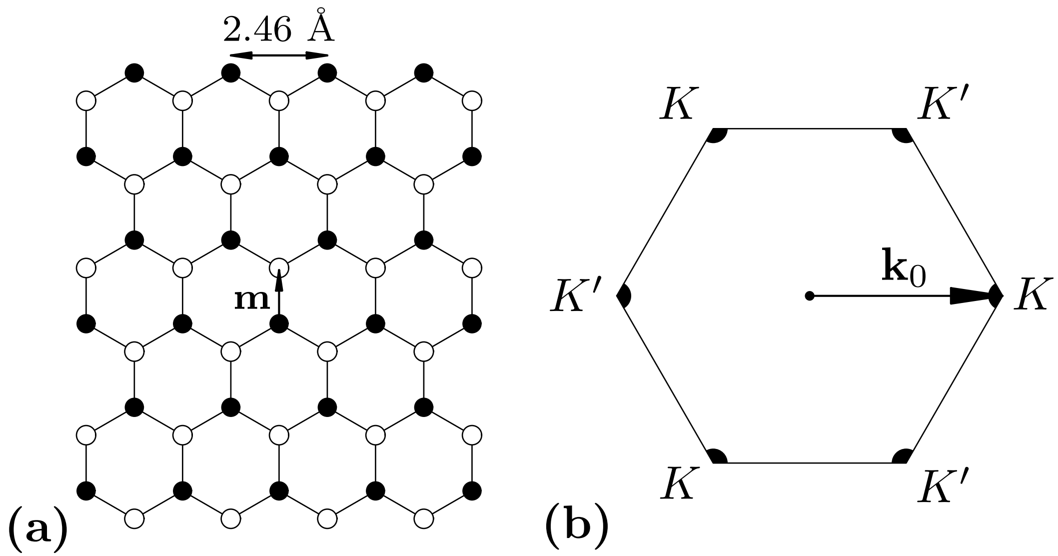
- disordered p-wave superconductor (class D):
  neither charge nor spin conserved; only energy conservation
  ⟶ TQH plateau transition: thermal Hall conductivity quantized
- very rich phase diagram: insulator, quantized Hall, and metallic phases
  ⟶ both QH and Anderson (metal-insulator) transitions
  ⟶ multicritical point

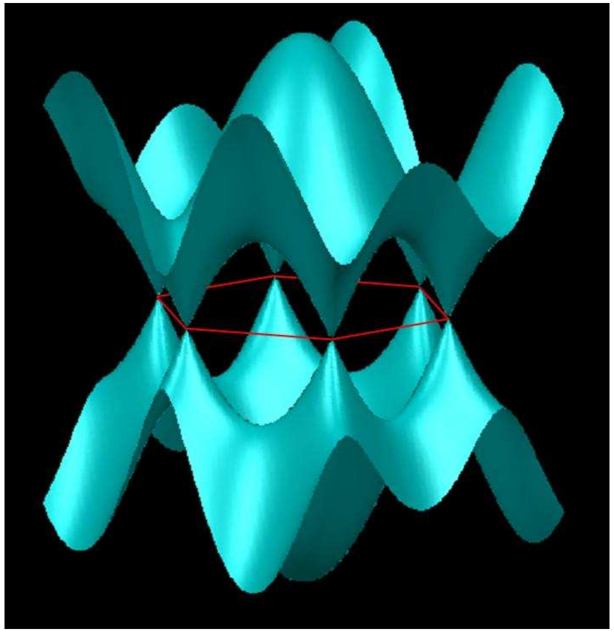




Senthil, Fisher'00; Bocquet et al '00; Read, Green'00, Read, Ludwig'01; Chalker et al '02; Mildenberger, Evers, Narayanan, Chalker, ADM'07

Graphene: 2D massless Dirac fermions





The gap vanishes at 2 points $\boldsymbol{K}$, $\boldsymbol{K'}$ (valleys)

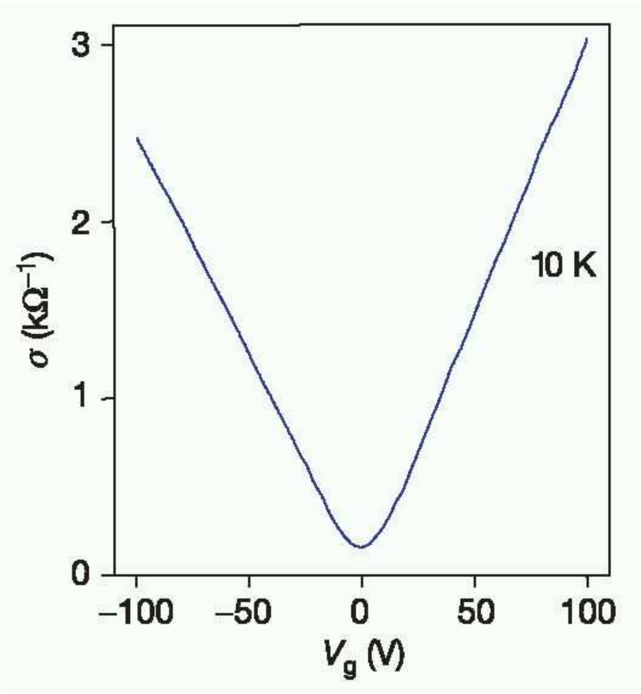
In the vicinity of $\boldsymbol{K}$, $\boldsymbol{K'}$ the spectrum is of massless Dirac-fermion type:

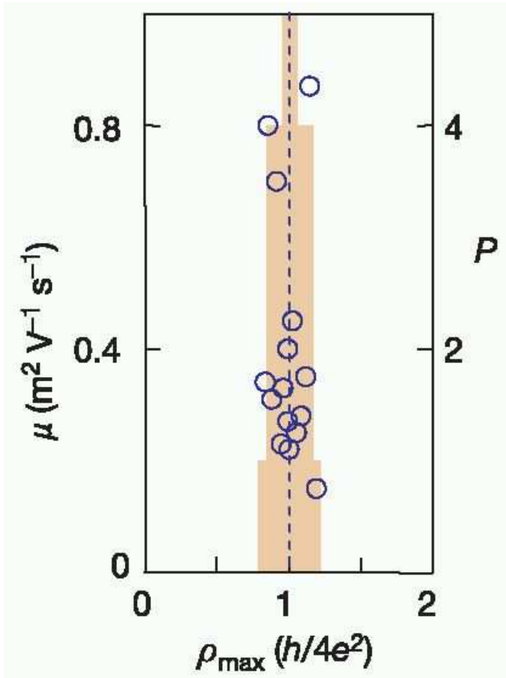
$$\boldsymbol{H}_K = \boldsymbol{v}_0(\boldsymbol{k}_x\boldsymbol{\sigma}_x + \boldsymbol{k}_y\boldsymbol{\sigma}_y), \qquad \boldsymbol{H}_{K'} = \boldsymbol{v}_0(-\boldsymbol{k}_x\boldsymbol{\sigma}_x + \boldsymbol{k}_y\boldsymbol{\sigma}_y)$$

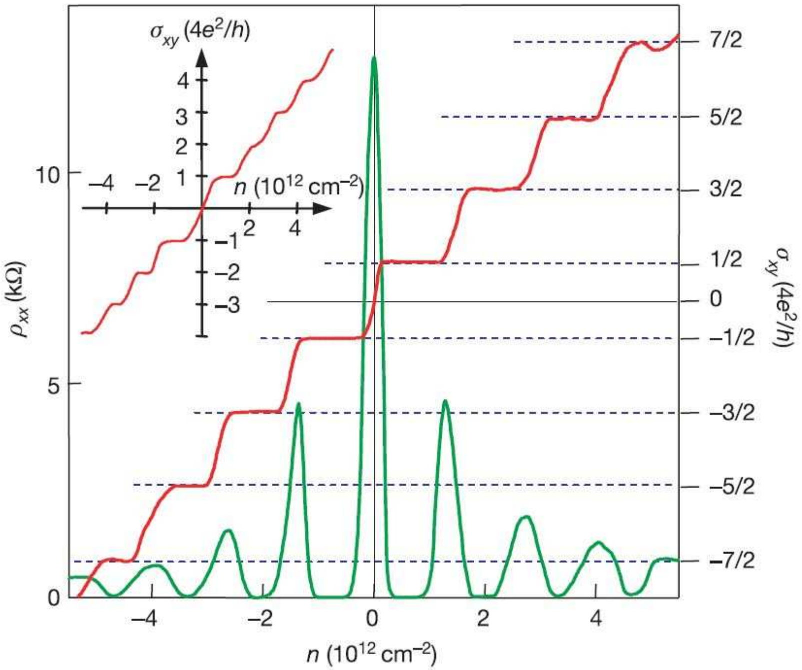
$v_0 \simeq 10^8$ cm/s – effective “light velocity”, sublattice space $\longrightarrow$ isospin

Graphene: Transport experiments

Novoselov, Geim et al; Zhang, Tan, Stormer, and Kim; Nature 2005







- minimal conductivity $\approx e^2/h$ per spin/valley,

almost T-independent in the range $1 \div 300$ K. Quantum criticality?

- anomalous, odd-integer Quantum Hall Effect $\sigma_{xy} = (2n+1) \times 2e^2/h$
- Quantum Hall Effect up to room temperature

(semiconductor record: 30 K)

Clean graphene: basis and notations

Hamiltonian $\longrightarrow$ 4×4 matrix operating in:

AB space of the two sublattices ($\boldsymbol{\sigma}$ Pauli matrices),

K–K' space of the valleys ($\boldsymbol{\tau}$ Pauli matrices).

Four-component wave function is chosen as

$$\Psi = \{\phi_{AK}, \phi_{BK}, \phi_{BK'}, \phi_{AK'}\}^T.$$

Hamiltonian in this basis has the form:

$$H = v_0 \tau_3 \boldsymbol{\sigma} \mathbf{k}.$$

Green function:

$$G_0^{R(A)}(\varepsilon, \mathbf{k}) = \frac{\varepsilon + v_0 \tau_3 \boldsymbol{\sigma}\mathbf{k}}{(\varepsilon \pm i0)^2 - v_0^2 k^2}.$$

Clean graphene: symmetries

Space of valleys K–K': Isospin $\Lambda_x = \sigma_3\tau_1$, $\Lambda_y = \sigma_3\tau_2$, $\Lambda_z = \sigma_0\tau_3$.

Time inversion

T_0 : $H = \sigma_1\tau_1 H^T \sigma_1\tau_1$

Chirality

C_0 : $H = -\sigma_3\tau_0 H \sigma_3\tau_0$

Combinations with $\Lambda_{x,y,z}$

T_x : $H = \sigma_2\tau_0 H^T \sigma_2\tau_0$

T_y : $H = \sigma_2\tau_3 H^T \sigma_2\tau_3$

T_z : $H = \sigma_1\tau_2 H^T \sigma_1\tau_2$

C_x : $H = -\sigma_0\tau_1 H \sigma_0\tau_1$

C_y : $H = -\sigma_0\tau_2 H \sigma_0\tau_2$

C_z : $H = -\sigma_3\tau_3 H \sigma_3\tau_3$

Spatial isotropy $\Rightarrow$ $T_{x,y}$ and $C_{x,y}$ occur simultaneously $\Rightarrow$ $T_\perp$ and $C_\perp$

Types of chiral disorder

(i) bond disorder: randomness in hopping elements t_{ij} (C_z-symmetry)

(ii) infinitely strong on-site impurities – unitary limit:
all bonds adjacent to the impurity are effectively cut
= bond disorder (C_z-symmetry)

(iii) dislocations: random non-Abelian gauge field (C_0-symmetry)

(iv) random magnetic field, ripples (all four symmetries $C_{0,x,y,z}$)

Symmetries of various types of disorder in graphene

		$\Lambda_\perp$	Λ_z	T_0	$T_\perp$	T_z	C_0	$C_\perp$	C_z	CT_0	$CT_\perp$	CT_z
$\sigma_0\tau_0$	α_0	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
$\sigma_{\{1,2\}}\tau_{\{1,2\}}$	$\beta_\perp$	$-$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$-$	$-$
$\sigma_{1,2}\tau_0$	$\gamma_\perp$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$+$
$\sigma_0\tau_{1,2}$	β_z	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$+$
$\sigma_3\tau_3$	γ_z	$-$	$+$	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$+$	$-$
$\sigma_3\tau_{1,2}$	β_0	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
$\sigma_0\tau_3$	γ_0	$-$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$+$
$\sigma_{1,2}\tau_3$	$\alpha_\perp$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$-$
$\sigma_3\tau_0$	α_z	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$

Related works:

S. Guruswamy, A. LeClair, and A.W.W. Ludwig, Nucl. Phys. B 583, 475 (2000)

E. McCann, K. Kechedzhi, V.I. Fal'ko, H. Suzuura, T. Ando, and B.L. Altshuler, PRL 97, 146805 (2006)

I.L. Aleiner and K.B. Efetov, PRL 97, 236801 (2006)

Symmetries of various types of disorder in graphene

		$\Lambda_\perp$	Λ_z	T_0	$T_\perp$	T_z	C_0	$C_\perp$	C_z	CT_0	$CT_\perp$	CT_z
$\sigma_0\tau_0$	α_0	+	+	+	+	+	−	−	−	−	−	−
$\sigma_{\{1,2\}}\tau_{\{1,2\}}$	$\beta_\perp$	−	−	+	−	−	+	−	−	+	−	−
$\sigma_{1,2}\tau_0$	$\gamma_\perp$	−	+	+	−	+	+	−	+	+	−	+
$\sigma_0\tau_{1,2}$	β_z	−	−	+	−	−	−	−	+	−	−	+
$\sigma_3\tau_3$	γ_z	−	+	+	−	+	−	+	−	−	+	−
$\sigma_3\tau_{1,2}$	β_0	−	−	−	−	+	−	−	+	+	−	−
$\sigma_0\tau_3$	γ_0	−	+	−	+	−	−	+	−	+	−	+
$\sigma_{1,2}\tau_3$	$\alpha_\perp$	+	+	−	−	−	+	+	+	−	−	−
$\sigma_3\tau_0$	α_z	+	+	−	−	−	−	−	−	+	+	+

Generic Gaussian disorder: thermodynamic and transport properties depend on

$$\alpha = \alpha_0 + \beta_0 + \gamma_0 + \alpha_\perp + \beta_\perp + \gamma_\perp + \alpha_z + \beta_z + \gamma_z$$

$$\alpha_{\rm tr} = \frac{1}{2}(\alpha_0 + \beta_0 + \gamma_0) + \alpha_\perp + \beta_\perp + \gamma_\perp + \frac{3}{2}(\alpha_z + \beta_z + \gamma_z)$$

2D disordered Dirac fermions: "ultraviolet RG" (history)

- random bond Ising model (random mass)

  V.S. Dotsenko and V.S. Dotsenko '83

- quantum Hall effect (random mass, scalar and vector potential)

  A.W.W. Ludwig, M.P.A. Fisher, R. Shankar, and G. Grinstein '94

- disordered d-wave superconductors

  A. A. Nersesyan, A. M. Tsvelik, and F. Wenger '94,'95
  M. Bocquet, D. Serban, and M.R. Zirnbauer '00
  A. Altland, B.D. Simons, and M.R. Zirnbauer '02

- Dirac fermions + C_z chiral disorder (e.g., random-field XY model)

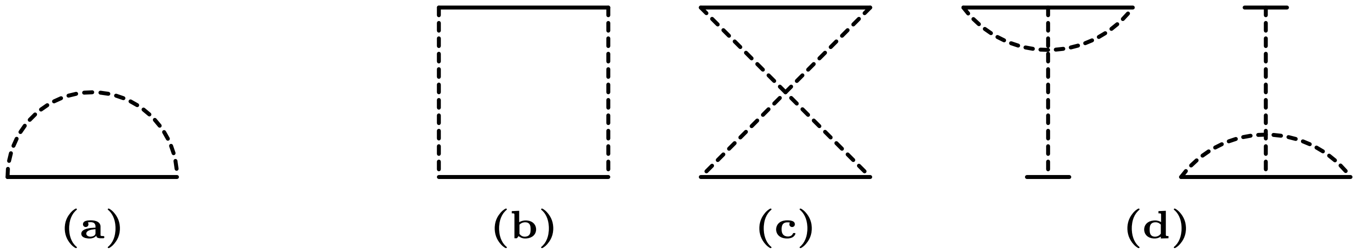
  S. Guruswamy, A. LeClair, and A.W.W. Ludwig '00

- disordered graphene (T_0 invariance + some hierarchy of couplings)

  I.L. Aleiner and K.B. Efetov '06.

Ultraviolet (Ballistic) Renormalization Group

v_0 =const, ε and disorder flow:



RG stops at length L_ε where ε reaches UV cutoff Δ

Logarithmic renormalization of disorder $\alpha_{\rm tr}$

$$\longrightarrow \qquad \sigma(\varepsilon) = \frac{e^2}{\pi^2 \alpha_{\rm tr}(L_\varepsilon)}$$

2D disordered Dirac fermions: complete one-loop RG

$$\frac{d\alpha_0}{d\log L} = 2\alpha_0(\alpha_0 + \beta_0 + \gamma_0 + \alpha_\perp + \beta_\perp + \gamma_\perp + \alpha_z + \beta_z + \gamma_z) + 2\alpha_\perp\alpha_z + \beta_\perp\beta_z + 2\gamma_\perp\gamma_z,$$

$$\frac{d\alpha_\perp}{d\log L} = 2(2\alpha_0\alpha_z + \beta_0\beta_z + 2\gamma_0\gamma_z),$$

$$\frac{d\alpha_z}{d\log L} = -2\alpha_z(\alpha_0 + \beta_0 + \gamma_0 - \alpha_\perp - \beta_\perp - \gamma_\perp + \alpha_z + \beta_z + \gamma_z) + 2\alpha_0\alpha_\perp + \beta_0\beta_\perp + 2\gamma_0\gamma_\perp,$$

$$\frac{d\beta_0}{d\log L} = 2[\beta_0(\alpha_0 - \gamma_0 + \alpha_\perp + \alpha_z - \gamma_z) + \alpha_\perp\beta_z + \alpha_z\beta_\perp + \beta_\perp\gamma_0],$$

$$\frac{d\beta_\perp}{d\log L} = 4(\alpha_0\beta_z + \alpha_z\beta_0 + \beta_0\gamma_0 + \beta_\perp\gamma_\perp + \beta_z\gamma_z),$$

$$\frac{d\beta_z}{d\log L} = 2[-\beta_z(\alpha_0 - \gamma_0 - \alpha_\perp + \alpha_z - \gamma_z) + \alpha_0\beta_\perp + \alpha_\perp\beta_0 + \beta_\perp\gamma_z],$$

$$\frac{d\gamma_0}{d\log L} = 2\gamma_0(\alpha_0 - \beta_0 + \gamma_0 + \alpha_\perp - \beta_\perp + \gamma_\perp + \alpha_z - \beta_z + \gamma_z) + 2\alpha_\perp\gamma_z + 2\alpha_z\gamma_\perp + \beta_0\beta_\perp,$$

$$\frac{d\gamma_\perp}{d\log L} = 4\alpha_0\gamma_z + 4\alpha_z\gamma_0 + \beta_0^2 + \beta_\perp^2 + \beta_z^2,$$

$$\frac{d\gamma_z}{d\log L} = -2\gamma_z(\alpha_0 - \alpha_\perp + \alpha_z - \beta_0 + \beta_\perp - \beta_z + \gamma_0 - \gamma_\perp + \gamma_z) + 2\alpha_0\gamma_\perp + 2\alpha_\perp\gamma_0 + \beta_\perp\beta_z.$$

$$\frac{d\varepsilon}{d\log L} = \varepsilon(1 + \alpha_0 + \beta_0 + \gamma_0 + \alpha_\perp + \beta_\perp + \gamma_\perp + \alpha_z + \beta_z + \gamma_z).$$

Conductivity at $\mu = 0$

For generic disorder, the Drude result $\sigma = 4 \times e^2/\pi h$ at $\mu = 0$ does not make much sense: Anderson localization will drive $\sigma \to 0$.

Experiment: $\sigma \approx 4 \times e^2/h$ independent of T

Quantum criticality ?

Can one have finite σ ?

Yes, if disorder either

(i) preserves one of chiral symmetries

or

(ii) is of long-range character (does not mix the valleys)

Critical states of disordered graphene

Disorder	Symmetries	Class	Conductivity	QHE
Vacancies	C_z, T_0	BDI	$\approx 4e^2/\pi h$	normal
Vacancies + RMF	C_z	AIII	$\approx 4e^2/\pi h$	normal
$\sigma_3\tau_{1,2}$ disorder	C_z, T_z	CII	$\approx 4e^2/\pi h$	normal
Dislocations	C_0, T_0	CI	$4e^2/\pi h$	chiral
Dislocations + RMF	C_0	AIII	$4e^2/\pi h$	chiral
Ripples, RMF	Λ_z, C_0	2×AIII	$4e^2/\pi h$	odd-chiral
Charged impurities	Λ_z, $T_\perp$	2×AII	$4\sigma_{Sp}^{**}$ or ∞	odd
random Dirac mass: $\sigma_3\tau_{0,3}$	Λ_z, $CT_\perp$	2×D	$4e^2/\pi h$	odd
Charged imp. + (RMF, ripples)	Λ_z	2×A	$4\sigma_U^*$	odd

C_0-chirality $H = -\sigma_3 H \sigma_3$: symmetry consideration

Disorder structures: $\sigma_{1,2}\tau_3, \quad \sigma_{1,2}\tau_{1,2}, \quad \sigma_{1,2}\tau_0$

Couplings: $\alpha_\perp$ (breaks T_0), $\beta_\perp$ (preserves T_0), $\gamma_\perp$ (preserves T_0)

Symmetry classes: C_0 + no T_0 = AIII (ChUE); $C_0 + T_0 =$ CI (BdG)

Generic C_0-disorder $\Rightarrow$ Non-Abelian vector potential problem

Nersesyan, Tsvelik, and Wenger '94,'95; Fendley and Konik '00; Ludwig '00

RG equations *(cf. Altland, Simons, and Zirnbauer '02)*:

$$\frac{\partial \alpha_\perp}{\partial \log L} = 0, \qquad \frac{\partial \beta_\perp}{\partial \log L} = 4\beta_\perp\gamma_\perp, \qquad \frac{\partial \gamma_\perp}{\partial \log L} = \beta_\perp^2.$$

DoS: $\rho(\varepsilon) \propto |\varepsilon|^{1/7}$ (generic C_0); $\rho(\varepsilon) \propto |\varepsilon|^{(1-\alpha_\perp)/(1+\alpha_\perp)}$ ($\alpha_\perp$ only)

Conductivity at $\mu = 0$: C_0-chiral disorder

Current operator $\quad \mathbf{j} = e v_0 \tau_3 \boldsymbol{\sigma}$

relation between G^R and G^A $\quad$ & $\quad \sigma_3 j^x = i j^y, \;\; \sigma_3 j^y = -i j^x,$

$\longrightarrow$ transform the conductivity at $\mu = 0$ to $RR + AA$ form:

$$\sigma^{xx} = -\frac{1}{\pi} \sum_{\alpha = x,y} \int d^2(r - r') \, \mathrm{Tr} \Big[j^\alpha G^R(0; \mathrm{r}, \mathrm{r}') j^\alpha G^R(0; \mathrm{r}', \mathrm{r}) \Big] \quad \equiv \sigma_{RR}.$$

Gauge invariance: $\quad \mathrm{p} \;\to \mathrm{p} + e\mathrm{A}$ $\quad$ constant vector potential

$$\sigma_{RR} = -\frac{2}{\pi} \frac{\partial^2}{\partial A^2} \, \mathrm{Tr} \;\; \ln G^R \qquad \longrightarrow \qquad \sigma_{RR} = 0 \quad (?!)$$

But: contribution with no impurity lines $\longrightarrow$ anomaly:

UV divergence $\Rightarrow$ shift of p is not legitimate *(cf. Schwinger model '62)*.

Universal conductivity at $\mu = 0$ for C_0-chiral disorder

Calculate explicitly (δ – infinitesimal ImΣ)

$$\sigma = -\frac{8e^2 v_0^2}{\pi\hbar} \int \frac{d^2k}{(2\pi)^2} \frac{\delta^2}{(\delta^2 + v_0^2 k^2)^2} = \frac{2e^2}{\pi^2\hbar} = \frac{4}{\pi}\frac{e^2}{h}$$

for C_0-chiral disorder $\sigma(\mu = 0)$ does not depend on disorder strength

Alternative derivation: use Ward identity

$$-ie(\mathrm{r} - \mathrm{r}')G^R(0; \mathrm{r}, \mathrm{r}') = [G^R \mathrm{j} G^R](0; \mathrm{r}, \mathrm{r}')$$

and integrate by parts $\longrightarrow$ only surface contribution remains:

$$\sigma = -\frac{ev_0}{4\pi^3} \oint d\mathrm{k}_n \, \mathrm{Tr}\left[\mathrm{j}\, G^R(\mathrm{k})\right] = \frac{e^2}{\pi^3\hbar} \oint \frac{d\mathrm{k}_n \mathrm{k}}{k^2} = \frac{4}{\pi}\frac{e^2}{h}$$

Related works: *Ludwig, Fisher, Shankar, Grinstein '94; Tsvelik '95*

C_z chirality: $\quad H = -\sigma_3\tau_3 H \sigma_3\tau_3$

Disorder structures: $\sigma_{1,2}\tau_0, \quad \sigma_0\tau_{1,2}, \quad \sigma_{1,2}\tau_3, \quad \sigma_3\tau_{1,2}$

Couplings:

$\gamma_\perp$ (preserves $T_{0,z}$), $\quad \beta_z$ (breaks T_z), $\quad \beta_0$ (breaks T_0), $\quad \alpha_\perp$ (breaks $T_{0,z}$)

Symmetry classes: $\quad C_z$ + no $T_{0,z}$ = AIII (ChUE);

$C_z + T_0$ = BDI (ChOE); $\quad C_z + T_z$ = CII (ChSE)

Related works: *Gade, Wegner '91; Hikami, Shirai, Wegner '93; Guruswamy, LeClair, Ludwig '00; Ryu, Mudry, Furusaki, Ludwig '07*

RG equations:

$$\frac{\partial \alpha_\perp}{\partial \log L} = 2\beta_0\beta_z, \qquad \frac{\partial \beta_0}{\partial \log L} = \frac{\partial \beta_z}{\partial \log L} = 2\alpha_\perp(\beta_0 + \beta_z), \qquad \frac{\partial \gamma_\perp}{\partial \log L} = \beta_0^2 + \beta_z^2$$

DoS: $\quad \rho(\varepsilon) \propto \varepsilon^{-1} \exp(-\#|\log \varepsilon|^{2/3})$

Conductivity:

$$\sigma = \frac{4e^2}{\pi h}\left[1 + \frac{1}{4}(\beta_0 - \beta_z)^2 + \cdots\right]$$

Long-range disorder

Smooth random potential does not scatter between valleys

Reduced Hamiltonian: $H = v_0 \boldsymbol{\sigma} \mathbf{k} + \sigma_\mu V_\mu(\mathbf{r})$

Ludwig, Fisher, Shankar, Grinstein '94; Ostrovsky, Gornyi, ADM '06-07

Disorder couplings: $\alpha_0 = \frac{\langle V_0^2 \rangle}{2\pi v_0^2}, \qquad \alpha_\perp = \frac{\langle V_x^2 + V_y^2 \rangle}{2\pi v_0^2}, \qquad \alpha_z = \frac{\langle V_z^2 \rangle}{2\pi v_0^2}$

Symmetries:

- α_0 disorder $\Rightarrow$ T-invariance $H = \sigma_y H^T \sigma_y$ $\Rightarrow$ AII (GSE)
- $\alpha_\perp$ disorder $\Rightarrow$ C-invariance $H = -\sigma_z H \sigma_z$ $\Rightarrow$ AIII (ChUE)
- α_z disorder $\Rightarrow$ CT-invariance $H = -\sigma_x H^T \sigma_x$ $\Rightarrow$ D (BdG)
- generic long-range disorder $\Rightarrow$ A (GUE)

Ultraviolet RG:

$$\frac{\partial \alpha_0}{\partial \ln L} = 2(\alpha_0 + \alpha_z)(\alpha_\perp + \alpha_0), \qquad \frac{\partial \alpha_\perp}{\partial \ln L} = 4\alpha_0\alpha_z, \qquad \frac{\partial \alpha_z}{\partial \ln L} = 2(\alpha_0 + \alpha_z)(\alpha_\perp - \alpha_z)$$

Long-range disorder (cont'd)

- **Class D** (random mass):

Disorder is marginally irrelevant $\Longrightarrow$ diffusion never occurs

DoS: $\rho(\varepsilon) = \frac{\varepsilon}{\pi v_0^2} 2\alpha_z \log \frac{\Delta}{\varepsilon}$ **Conductivity:** $\sigma = \frac{4e^2}{\pi h}$

- **Class AIII** (random vector potential):

C_0/C_z chiral disorder; considered above

DOS: $\rho(\varepsilon) \propto |\varepsilon|^{(1-\alpha_\perp)/(1+\alpha_\perp)}$ **Conductivity:** $\sigma = \frac{4e^2}{\pi h}$

Long-range disorder: unitary symmetry

Generic long-range disorder breaks all symmetries $\Longrightarrow$ class A (GUE)

Effective infrared theory is Pruisken's unitary sigma model with θ-term:

$$S[Q] = \frac{1}{4}\,\mathrm{Str}\left[-\frac{\sigma_{xx}}{2}(\nabla Q)^2 + \left(\sigma_{xy} + \frac{1}{2}\right) Q\nabla_x Q \nabla_y Q\right] \Rightarrow -\frac{\sigma_{xx}}{8}\,\mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$$

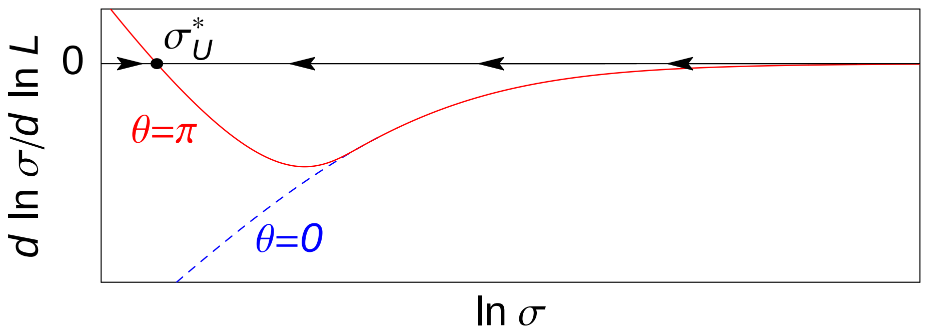
Compact (FF) sector of the model: $Q_{\mathrm{FF}} \in \mathcal{M}_{\mathrm{F}} = \dfrac{U(2n)}{U(n)\times U(n)}$

Topological term takes values $N[Q] \in \pi_2(\mathcal{M}_{\mathrm{F}}) = \mathbb{Z}$

Vacuum angle $\theta = \pi$ in the absence of magnetic field due to anomaly

$\Longrightarrow$ Quantum Hall critical point

$$\sigma = 4\sigma_U^* \approx \frac{2e^2}{h}$$



Long-range disorder: symplectic symmetry

Diagonal disorder α_0 (charged impurities) preserves T-inversion symmetry

$\Longrightarrow$ class AII (GSE)

Partition function is real $\Longrightarrow$ $\operatorname{Im} S = 0$ or π

Compact sector: $$Q_{\mathrm{FF}} \in \mathcal{M}_{\mathrm{F}} = \frac{O(4n)}{O(2n) \times O(2n)}$$

$$\Longrightarrow \quad \pi_2(\mathcal{M}_{\mathrm{F}}) = \begin{cases} \mathbb{Z} \times \mathbb{Z}, & n = 1; \\ \mathbb{Z}_2, & n \geq 2 \end{cases}$$

At $n = 1$ $\quad \mathcal{M}_{\mathrm{F}} = S^2 \times S^2/\mathbb{Z}_2 \approx$ [Cooperons] $\times$ [diffusons]

$$\Longrightarrow \quad \operatorname{Im} S = \theta_c N_c[Q] + \theta_d N_d[Q]$$

T-invariance $\longrightarrow$ $\theta_c = \theta_d = 0$ or π $\qquad$ Anomaly $\longrightarrow$ $\theta_{c,d} = \pi$

At $n \geq 2$ we use $\mathcal{M}_{\mathrm{F}}|_{n=1} \subset \mathcal{M}_{\mathrm{F}}|_{n\geq 2}$ $\Longrightarrow$ $\operatorname{Im} S = \pi N[Q]$

Symplectic sigma-model with $\theta = \pi$ term: $S[Q] = -\frac{\sigma_{xx}}{16} \operatorname{Str}(\nabla Q)^2 + i\pi N[Q]$

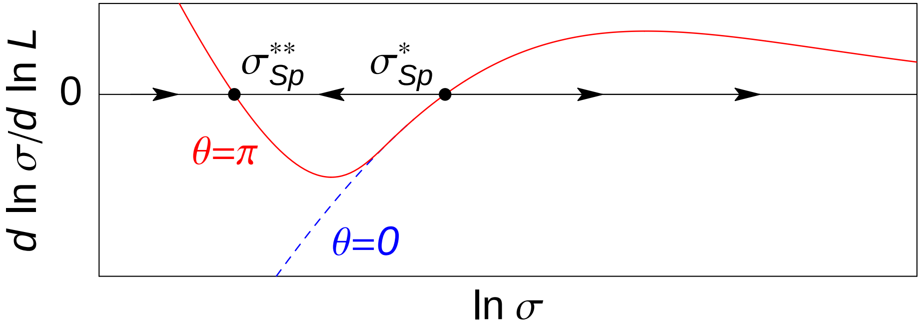
Long-range disorder: symplectic symmetry (cont'd)

Symplectic sigma-model with $\theta = \pi$ term: $S[Q] = -\frac{\sigma_{xx}}{16}\,\mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$

Similar to Pruisken sigma-model of QHE at criticality

Instantons suppress localization

$\Longrightarrow$ Novel critical point ?



Conductivity: $\sigma = 4\sigma^{**}_{Sp}$

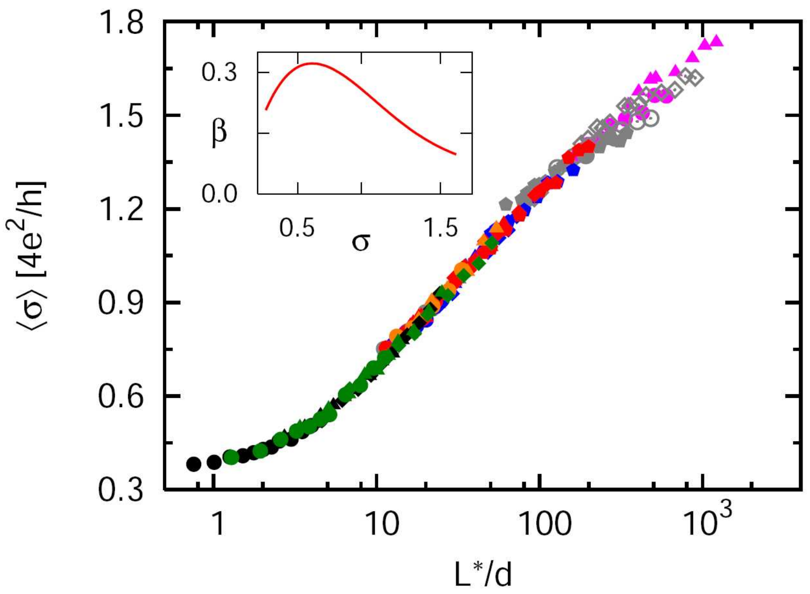
Alternative scenario: β function everywhere positive, $\sigma \to \infty$

Numerics needed

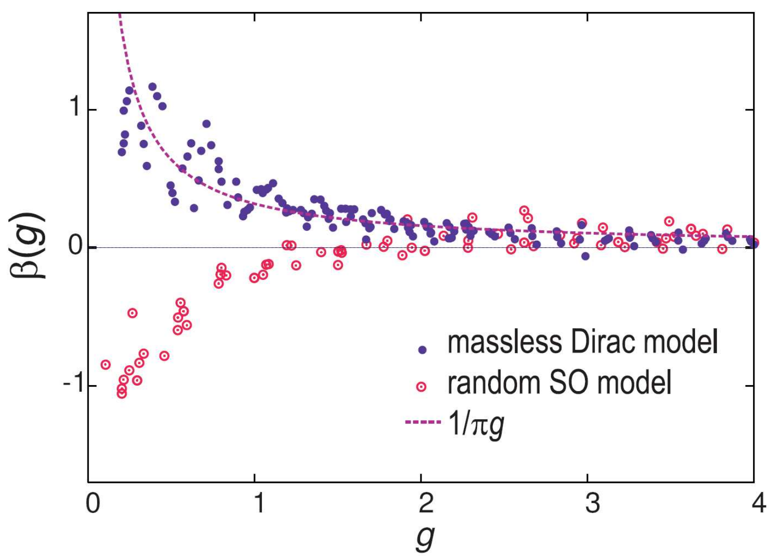
Long-range potential disorder: numerics

Recent numerics confirm the absence of localization but the new critical point is not found.

Bardarson, Tworzydło, Brouwer, Beenakker
arXiv:0705.0886



Nomura, Koshino, Ryu
arXiv:0705.1607



Comments

- **If an additional attractive fixed point exists, different microscopic models may flow into different fixed points**

- **Coulomb interaction (Altshuler-Aronov, Finkelstein) favors a novel fixed point**

- **Novel fixed point (if/when exists) is expected to describe the Quantum Spin Hall transition (in the presence of disorder)**

Odd quantum Hall effect

Decoupled valleys + magnetic field $\Longrightarrow$

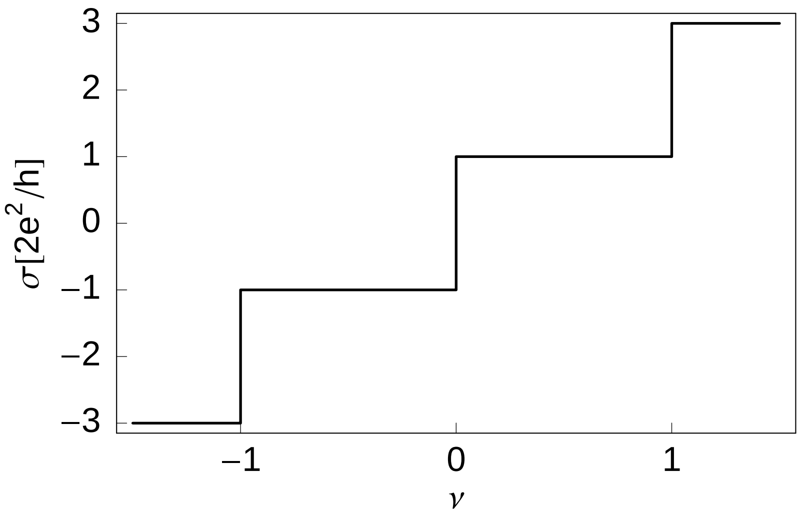
unitary sigma model with anomalous topological term:

$$S[Q] = \frac{1}{4}\mathrm{Str}\left[-\frac{\sigma_{xx}}{2}(\nabla Q)^2 + \left(\sigma_{xy} + \frac{1}{2}\right) Q\nabla_x Q \nabla_y Q\right] \Longrightarrow \text{odd-integer QHE}$$

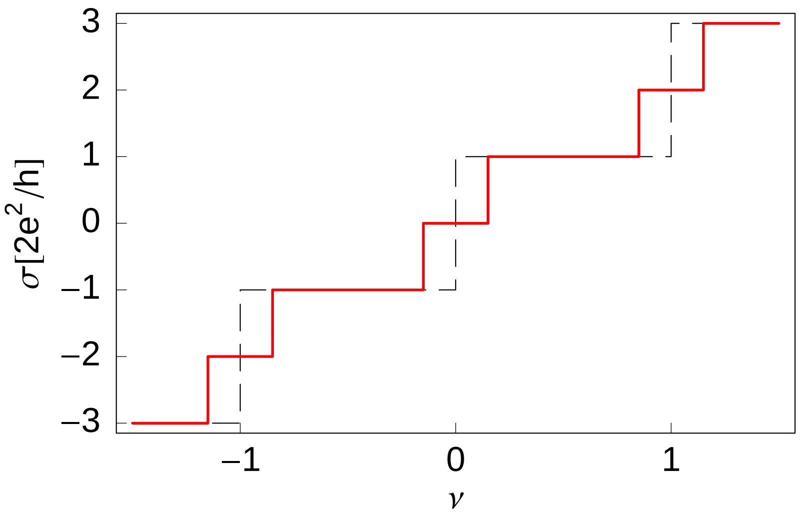
generic (valley-mixing) disorder $\Longrightarrow$ conventional IQHE

weakly valley-mixing disorder $\Longrightarrow$ even plateaus narrow, emerge at low T

Decoupled valleys



Weakly mixed valleys



Estimate: $T_{\mathrm{mix}} \sim 10 - 50\mathrm{mK}$

Chiral quantum Hall effect

C_0**-chiral disorder** $\iff$ **random vector potential**

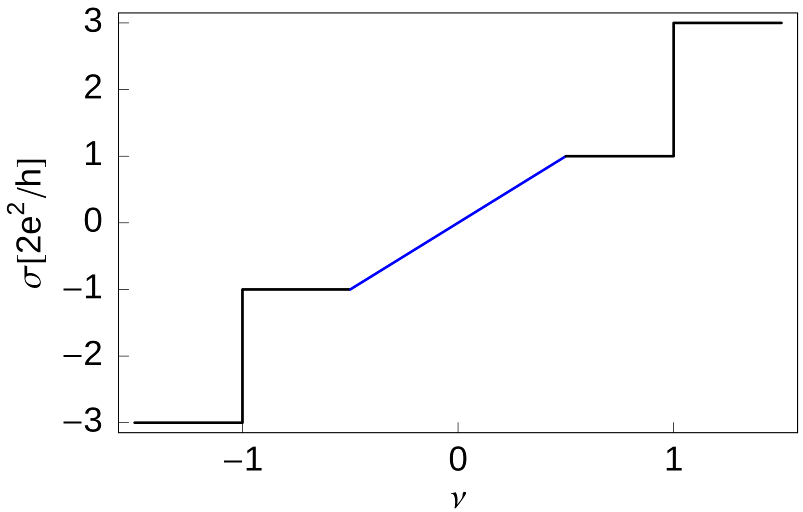
Atiyah-Singer theorem:

In magnetic field, zeroth Landau level remains degenerate!!!

(Aharonov and Casher '79)

Within zeroth Landau level Hall effect is classical

Decoupled valleys (ripples)



Weakly mixed valleys (dislocations)